ALTERNATING CURRENT

Until now, we have studied only circuits with direct current (dc) which flows only in one direction. the primary source of emf in such circuit is a battery. When a resistance is connected across the terminals of the battery, a current is established in the circuits, which flows in a unique direction from the positive terminal to the negative terminal via the external resistance.

But most of the electric power generated and used in the world is in the form of alternating current (ac), the magnitude of which changes continuously with time and direction is reversed periodically as shown in figure and it is given by

$$i = i_o \sin(\omega t + \phi) \qquad (1)$$

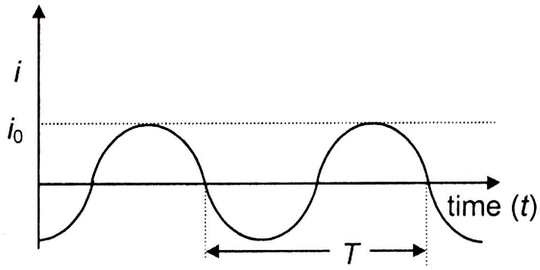


Fig. (a)

Here i is instantaneous value of currnet i.e., magnitude of current at any instant of time and i_o is the maximum value of current which is called peak current or the current amplitude and the current repeats its value after each time interval T = $2\pi / \omega$ as shown in figure. This time interval is called the time period and ω is angular frequency which is equal to 2π times of frequency f.

$$\omega = 2\pi f$$

The current is positive for half the time period and negative for remaining half period. It means direction of current is reversed after each half time period. The frequency of ac in India is 50 Hz.
An alternating voltage is given by

$$V = V_o \sin(\omega t + \phi) \qquad (2)$$

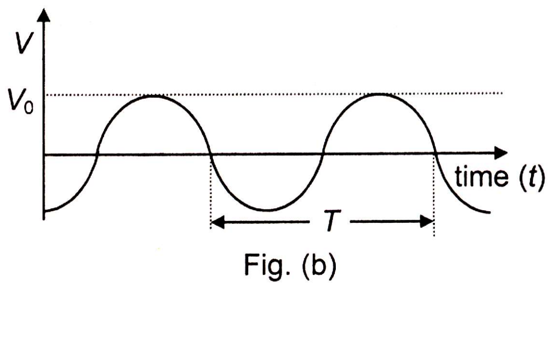
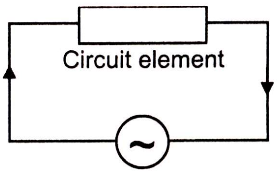


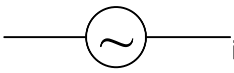
Fig. (b)

It is also varies alternatively as shown in the figure (b), where V is instantaneous voltage and V_o is peak voltage. It is produced by ac generator also called as ac dynamo.

AC circuit:

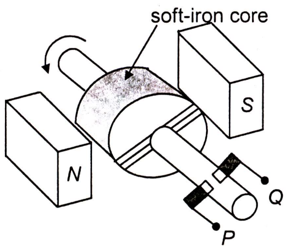
An ac circuit consists of circuit element i.e., resistor, capacitor, inductor or any combination of these and a generator that provides the alternating current as shown in figure. The ac source is represented by



symbol  in the circuit.

AC Generator

The basic principle of the ac generator is a direct consequence of Faraday's laws of electromagnetic induction. When a coil of N turns and area of cross section A is rotated in a uniform magnetic field B with constant angular velocity ω as shown in figure, a sinusoidal voltage (emf) is induced in the coil.



Suppose the plane of the coil at t = 0 is perpendicular to the magnetic field and in time t, it rotates through an angle θ,

Therefore, flux through the coil at time t is

$$\phi = NBA\cos\theta = NBA\cos\omega t$$

$$\frac{d\phi}{dt} = -NBA\omega\sin\omega t \qquad (3)$$

The emf induced in the coil is,

$$\varepsilon = \frac{-d\phi}{dt} = NBA\omega\sin\omega t$$

$$\varepsilon = \varepsilon_o\sin\omega t, \text{ where } \varepsilon_o = NBA\omega$$

ε_o is maximum value of emf, which is called peak emf or voltage amplitude and current,

$$i = \frac{\varepsilon}{R} = \frac{\varepsilon_o}{R}\sin\omega t = i_o\sin\omega t \qquad (4)$$

where i_o is the PEAK value of current.

AVERAGE AND RMS VALUE OF ALTERNATING CURRENT

Average Current (Mean Current)

As we know an alternating current is given by

$i = i_o \sin(\omega t + \phi)$ (i)

The mean or average value of ac over any time T is given by

$$i_{avg} = \frac{\int_0^T i\,dt}{\int_0^T dt}$$

Using equation (i)

$$i_{avg} = \frac{\int_0^T i_o \sin(\omega t + \phi)\,dt}{\int_0^T dt}$$

In one complete cycle average current

$$i_{avg} = -\frac{i_o}{T}\left[\frac{\cos(\omega t + \phi)}{\omega}\right]_0^T$$

$$= -\frac{i_o}{T}\left[\frac{\cos(\omega t + \phi) - \cos\phi}{\omega}\right] = -\frac{i_o}{T}\left[\frac{\cos(2\pi + \phi) - \cos\phi}{\omega}\right] = 0 \qquad (\text{as } \omega T = 2\pi)$$

Since ac is positive during the first half cycle and negative during the other half cycle so i_{avg} will be zero for long time also. Hence the dc instrument will indicate zero deflection when connected to a branch carrying ac current. So it is defined for either positive half cycle or negative half cycle.

$$i_{avg} = \frac{\int_0^{T/2} i_o \sin(\omega t + \phi)}{\int_0^{T/2} dt} = \frac{2i_o}{\pi} = 0.637 i_o \qquad (5)$$

Similarly $V_{avg} = \frac{2V_o}{\pi} = 0.637 V_o$

This is also known as rectified average value of sinusoidal voltage.

RMS Value of Alternating Current

The notation rms refers to root mean square, which is given by square root of mean of square root.

i.e. $i_{rms} = \sqrt{i_{avg}^2}$

$$i_{avg}^2 = \frac{\int_0^T i^2 dt}{\int_0^T dt} = \frac{1}{T}\int_0^T i_o^2 \sin^2(\omega t + \phi)\,dt = \frac{i_o^2}{2T}\int_0^T [1 - \cos 2(\omega t + \phi)]\,dt$$

$$= \frac{i_o^2}{2T}\left[t - \frac{\sin 2(\omega t + \phi)}{2\omega}\right]_0^T = \frac{i_o^2}{2T}\left[T - \frac{\sin(4\pi + 2\phi) - \sin 2\phi}{2\omega}\right] = \frac{i_o^2}{2}$$

$$i_{rms} = \frac{i_o}{\sqrt{2}} \approx 0.707 i_o \qquad (7)$$

Similarly the rms voltage is given by $V_{rms} = \frac{V_o}{\sqrt{2}} \approx 0.707 V_o$ (8)

The significance of rms current and rms voltage may be shown by considering a resistance R carrying a current $i = i_o \sin(\omega t + \phi)$.

The voltage across the resistor will be

$$V_R = Ri = (i_o R)\sin(\omega t + \phi)$$

The thermal energy developed in the resistor during the time t to t + dt is

$$i^2 R dt = i_o^2 R \sin^2(\omega t + \phi) dt$$

The thermal energy developed in one time period is

$$U = \int_0^T i^2 R dt = R\int_0^T i_o^2 \sin^2(\omega t + \phi) dt = RT\left[\frac{1}{T}\int_0^T i_o^2 \sin^2(\omega t + \phi) dt\right] = i_{rms}^2 RT \qquad (9)$$

It means that the root mean square value of ac is that value of steady current, which would generate the same amount of heat in a given resistance in a given time.

So in ac circuits, current and ac voltage are measured in terms of their rms values. Like when we say that the house hold supply is 220 V ac it means the rms value is 220 V and peak value is $220\sqrt{2} = 311\text{V}$.

Illustration : 1

If the voltage in an ac circuit is represented by the equation $V = 220\sqrt{2}\sin(314t - \phi)$, calculate (a) peak and rms value of the voltage, (b) average voltage, (c) frequency of ac.

Solution

(a) For ac votlage, $V = V_o \sin(\omega t - \phi)$

The peak value, $V_o = 220\sqrt{2} = 311\text{V}$

The rms value of voltage, $V_{rms} = \frac{V_o}{\sqrt{2}}$; $V_{rms} = 220\,\text{V}$

(b) Average voltage in full cycle is zero. Average voltage in half cycle is

$$V_{avg} = \frac{2}{\pi}V_o = \frac{2}{\pi} \times 311 = 198.17\,\text{V}$$

(c) As $\omega = 2\pi f$, $2\pi f = 314$

i.e. $f = \frac{314}{2 \times \pi} = 50\,\text{Hz}$

Illustration : 2

The electric current in a circuit is given by $i = i_o\left(t/T\right)$ for some time. Calculate the rms current for the period t = 0 to t = T.

Solution

The mean square current is

$$\left(i^2\right)_{avg} = \frac{1}{T}\int_0^T i_o^2\left(t/T\right)^2 dt = \frac{i_o^2}{T^3}\int_0^T t^2 dt = \frac{i_o^2}{3}$$

Thus, the rms current is $i_{rms} = \sqrt{i^2_{avg}} = \frac{i_o}{\sqrt{3}}$

Illustration: 3

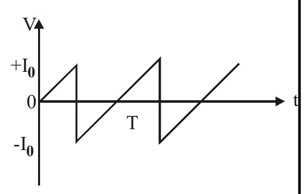
Find the average current in terms of I_0 for the waveform shown.

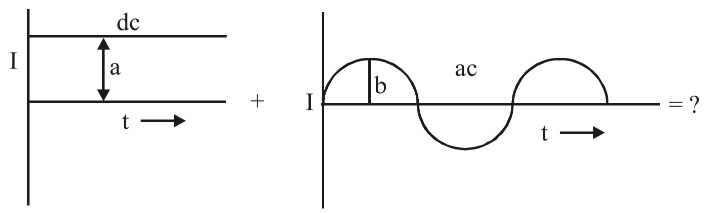
Solution: $I = 2I_0\frac{t}{T};\ \ 0 < t < \frac{T}{2}\ ;$ $\qquad I = 2I_0\left(\frac{t}{T} - 1\right)\ ;\ \frac{T}{2} < t < T$

$$I_{av} = \frac{2}{T}\int_0^T I.dt = \frac{2}{T}\left[\int_0^{T/2}\frac{2I_0 t}{T}dt\right] \qquad = \frac{2}{T^2}\left[\frac{2I_0T^2}{2\times 4}\right] = \frac{I_0}{2}$$

Illustration: 4

. If a direct current of value a ampere is superimposed on an alternating current I = b sin ωt flowing through a wire, what is the effective value of the resulting current in the circut ?





Solution

As current at any instant in the circuit will be,

$$i = i_{dc} + i_{ac} = a + b\sin\omega t \qquad \text{So } i_{eff} = \left[\frac{\int_0^T I^2 dt}{\int_0^T dt}\right]^{1/2} = \left[\frac{1}{T}\int_0^T\left(a + b\sin\omega t\right)^2 dt\right]^{1/2}$$

i.e. $\quad i_{eff} = \left[\frac{1}{T}\int_0^T\left(a^2 + 2ab\sin\omega t + b^2\sin^2\omega t\right)dt\right]^{1/2}$ but as

But as $\frac{1}{T}\int_0^T \sin\omega t dt = 0$ and $\frac{1}{T}\int_0^T \sin^2\omega t dt = \frac{1}{2}$ so $i_{eff} = \left[a^2 + \frac{1}{2}b^2\right]^{1/2}$

Phasors and Phasor Diagrams

In the study of AC circuits, we shall come across alternating voltages and currents which have the same frequency but differ in phase with each other. It is found that the study of AC circuit becomes simple, if alternating currents and voltages are treated as rotating vectors or more correctly as '**phasors**'. The phase angle between the two quantities is also represented in the vector diagram.

A diagram representing alternating voltage and current as vectors with the phase angle between them is known as **phasor diagram**.

e.g. $V = V_o \sin(\omega t)$

$$i = i_o \sin(\omega t + \phi)$$

where ϕ is the phase angle between alternating emf and current.

The instantaneous values V and i may be regarded as projections of V_o and i_o respectively, if V_o and i_o are treated as rotating vectors or more correctly as 'phasors'. A diagram representing alternating voltage and current as rotating vectors with the phase angle between them is known as a phasor diagram.

The phasor diagram of $V = V_o \sin\omega t$ and $i = i_o \sin(\omega t + \phi)$ is shown in figure (a).

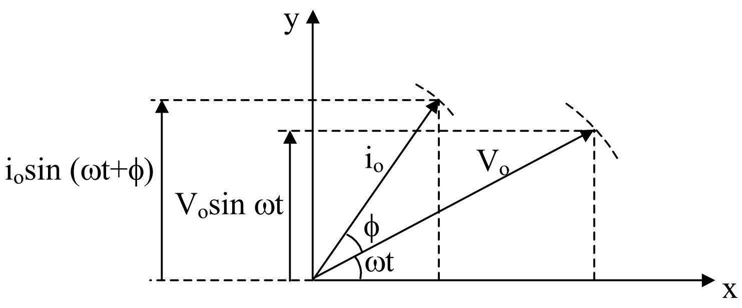


fig.(a)

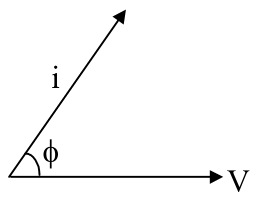


fig.(b)

If we are interested only in phase relationship, the phasor diagram may also be represented as in figure (b).

SERIES AC CIRCUIT

When Only Resistance is in an AC Cricuit

Consider a simple ac circuit consisting of a resistor of resistance R and an ac generator, as shown in the figure.

According to Kirchhoff's loop law at any instant, the algebraic sum of the potential difference around a closed loop in a circuit must be zero.

$$\varepsilon - V_R = 0$$

$$\varepsilon - i_R R = 0$$

$$\varepsilon_o \sin\omega t - i_R R = 0$$

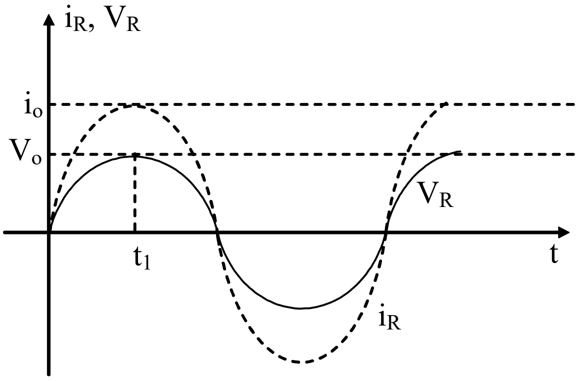
$$i_R = \frac{\varepsilon_o}{R}\sin\omega t = i_o \sin\omega t \qquad (1)$$

where i_o is the mximum current. $i_o = \dfrac{\varepsilon_o}{R}$

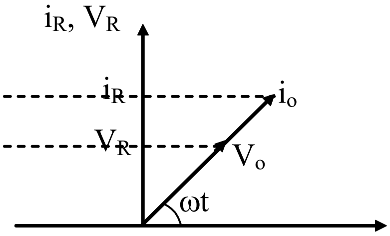
From the above equations, we see that the instantaneous voltage drop across the resistor is

$V_R = i_o R \sin \omega t$ (2)

We see in equation (1) and (2), i_R and V_R both vary as $\sin \omega t$ and reach their maximum values at the same time as shown in figure (a), they are said to be in phase. A phasor diagram is used to represent phase relationships. The lengths of the arrows correspond to V_o and i_o. The projections of the arrows onto the vertical axis give V_R and i_R. In case of the single-loop resistive circuit, the current and voltage phasors lie along the same line, as shown in figure (b), because **i_R and V_R are in phase**.



Wave diagram
Fig.(a)



Phasor diagram
Fig.(b)

WHEN ONLY INDUCTOR IS IN AN AC CIRCUIT

Now consider an ac circuit consisting only of an inductor of inductance L connected to the terminals of an ac generator, as shown in the figure. The induced emf across the inductor is given by Ldi/dt. On applying Kirchhoff's loop rule to the circuit

$$\varepsilon - V_L = 0 \Rightarrow \varepsilon - L\frac{di}{dt} = 0$$

When we rearrange this equation and substitute $\varepsilon = \varepsilon_o \sin \omega t,$ we get

$$\varepsilon = \varepsilon_o \sin \omega t, \qquad (3)$$

Integration of this expression gives the current as a function of time

$$i_L = \frac{\varepsilon_o}{L}\int \sin \omega t dt = -\frac{\varepsilon_o}{\omega L}\cos \omega t + C$$

For average value of current over one time period to be zero, C = 0.

$$\therefore \qquad i_L = -\frac{\varepsilon_o}{\omega L}\cos \omega t$$

When we use the trigonometric identity $\cos \omega t = -\sin\left(\omega t - \frac{\pi}{2}\right)$, we can express equation as

$$i_L = \frac{\varepsilon_o}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) \qquad (4)$$

From equation (4), we see that the current reaches its maximum values when $\cos \omega t = 1$.

$$i_o = \frac{\varepsilon_o}{\omega L} = \frac{\varepsilon_o}{X_L} \qquad (5)$$

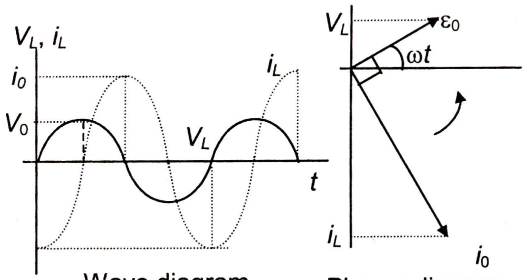
where the quantity X_L, called the inductive reactance, is

$$X_L = \omega L$$

The expression for the rms current is similar to equation (5), with ε_o replaced by ε_{rms}.
Inductive reactance, like resistance, has unit of ohm.

$$V_L = L\frac{di}{dt} = \varepsilon_o \sin\omega t = i_o X_L \sin\omega t$$

We can think equation (5) as Ohm's law for an inductive circuit.
On comparing result of equation (4) with equation (3), we can see that the current and voltage are out of phase with each other by $\pi/2$ rad or 90°. A plot of voltage and current versus time is givenin figure (a). The voltage reaches its maximum value one quarter of an oscillation period before the current reaches its maximum value. The corresponding phasor diagram for this circuit is shown in figure (b). Thus, we see that for a sinusoidal applied voltage, the current in an inductor always lags behind the voltage across the inductor by 90°.



Wave diagram
Fig. (a)

Phasor diagram
Fig. (b)

WHEN ONLY CAPACITOR IS IN AN AC CIRCUIT
Figure shows an ac circuit consisting of a capacitor of capacitance C connected across the terminals of an ac generator. On applying Kirchhoff's loop rule to this circuit, we get

$$\varepsilon - V_C = 0$$

$$V_C = \varepsilon = \varepsilon_o \sin\omega t \qquad (6)$$

where V_C is the instantaneous voltage drop across the capacitor. From the definition of capacitance, V_C = Q/C, and this value for V_C substituted into equation gives

$$Q = C\varepsilon_o \sin\omega t$$

Since $i = dQ/dt = C\varepsilon_o\omega \cos\omega t$
Here again we see that the current is not in phase with the voltage drop across the capacitor, given by equation (6). Using the trigonometric identity $\cos\omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$, we can express this equation in the alternative form

$$i_C = \omega C\varepsilon_o \sin\left(\omega t + \frac{\pi}{2}\right) \qquad (7)$$

From equation (7), we see that the current in tbe circuit reaches its maximum value when $\cos\omega t = 1$.

$$i_o = \omega C\varepsilon_o = \frac{\varepsilon_o}{X_C}$$

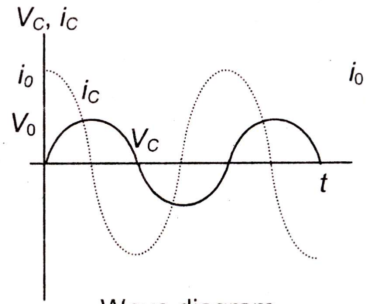
where X_C is called the capacitive reactance.

$$X_C = \frac{1}{\omega C}$$

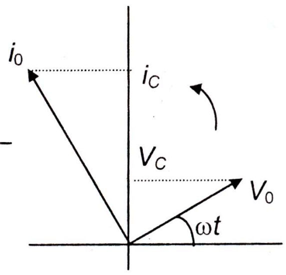
The SI unit of X_C is also ohm. The rms current is given by an expression similar to equation with V_o replaced by V_{rms}.
Combining equations (6) and (7), we can express the instantaneous voltage drop across the capacitor as

$$V_C = V_o \sin\omega t = i_o X_C \sin\omega t$$

Comparing the result of equation (7) with equation (6), we see that the current is $\pi/2$ rad = 90° out of phase with the voltage across the capacitor. A plot of current and voltage versus time, shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches its maximum value. The corresponding phasor diagram is shown in figure (b). Thus we see that for a sinusoidally applied emf, the current always leads the votlage across a capacitor by 90°.



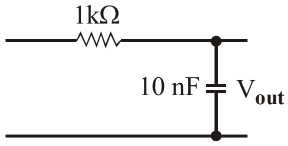
Wave diagram
Fig. (a)



Phasor diagram
Fig. (b)

Illustration: 5

If an input of 50 mV is applied as V_{in} then V_{out} at 100 kHz will be



Solution

$$V_0 = \frac{V_{in}}{|Z|}X_C = \frac{50\times 159}{\sqrt{1000^2 + 159^2}} \approx 7.9\,\text{mV} \quad ; \quad X_C = \frac{1}{C\omega} = \frac{1}{10^{-8}\times 2\pi\times 10^5} = \frac{10^{-3}}{2\pi} \approx 159\,\Omega$$

Illustration: 6

30.0 μF capacitor is connected to a 220 V, 50 Hz source. Find the capacitive reactance and the current (rms and peak) in the circuit. If the frequency is doubled, what happens to the capacitive reactance and the current.

Solution: The capacitive reactance is $X_C = \frac{1}{2\pi f C} = 106\Omega$

The rms current is $I_{rms} = \frac{V_{rms}}{X_C} = 2.08\text{A}$

The peak current is $I_m = \sqrt{2}I_{rms} = 2.96\text{A}$

This current oscillates between 2.96A and – 2.96A and is ahead of the voltage by 90°.
If the frequency is doubled, the capacitive reactance is halved and consequently, the current is doubled.

Vector Analysis (Phasor Algebra)

The complex quantities normally employed in ac circuit analysis, can be added and subtracted like coplanar vectors. Such coplanar vectors, which represent sinusoidally time varying quantities, are known as phasors.

A Cartesian form, a phasor A can be written as,

$$A = a + jb$$

where a is the x-component and b is the y-component of phasor A.

The magnitude of A is, $|A| = \sqrt{a^2 + b^2}$

and the angle between the direction of phasor A and the positive x-axis is

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

When a given phasor A, the direction of which is along the positive x-axis is multiplied by the operator j, a new phsor jA is obtained which will be 90° anticlockwise from A, i.e., along y-axis. If the operator j is multiplied now to the phasor jA, a new phasor j^2A is obtained which is along the negative x-axis and haivng same magnitude as of A. Thus,

$$j^2A = -A$$

$$j^2 = -1 \quad \text{or} \quad j = \sqrt{-1}$$

Now using the j operator, let us discuss different circuits of an ac.

Series L-R Circuit

Now consider an ac circuit consisting of a resistor of resistance R and an inductor of inductance L in series with an ac source generator.

Suppose in phasor diagram, current is taken along positive x-direction. The V_R is also also along positive x-direction and V_L along positive y-direction as we know potential difference across a resistance in ac is in phase with current and it leads in phase by 80° with current across the inductor, so we can write

$$V = V_R + jV_L = iR + j(iX_L) = iR + j(i\omega L) = iZ$$

Here, $Z = R + jX_L = R + j(\omega L)$ is called as impedance of the circuit. Impedance plays the same role in ac circuits as the ohmic resistance does in dc circuits. The modulus of impedance is,

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

The potential difference leads the current by an angle,

$$\phi = \tan^{-1}\left|\frac{V_L}{V_R}\right| = \tan^{-1}\left(\frac{X_L}{R}\right) \qquad \phi = \tan^{-1}\left(\frac{\omega L}{R}\right)$$

Illustration: 7

A 0.21 H inductor and a 12 ohm resistance are connected in series to a 220 V, 50 Hz ac source. Calculate the current in the circuit and the phase angle between the current and the source voltage.

Solution

Here $X_L = \omega L = 2\pi f L = 2\pi \times 50 \times 0.21 = 21\pi\Omega$

So, $Z = \sqrt{R^2 + X^2} = \sqrt{12^2 + (21\pi)^2} = \sqrt{144 + 4348}$

i.e., $Z = \sqrt{4492} \approx 67.02\Omega$

So (a) $I = \frac{V}{Z} = \frac{220}{67.02} = 3.28\,A$ and (b) $\phi = \tan^{-1}\left(\frac{X}{R}\right) = \tan^{-1}\left(\frac{21\pi}{12}\right) = \tan^{-1}(5.5) = 79.7^\circ$

i.e., the current will lag the applied voltage by 79.7° in phase.

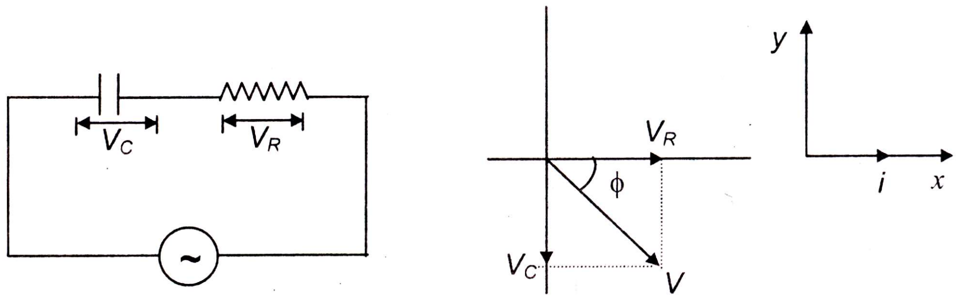
Series C-R Circuit

Now consider an ac circuit consisting of a resistor of resistance R and an capacitor of capacitance C in series with an ac source generator.

Suppose in phasor diagram current is taken along positive x-direction. Then V_R is along negative y-direction as potential difference across a capacitor in ac lags in phase by 90° with the current in the circuit. So we can write,

$$V = V_R - jV_C = iR - j(iX_C) = iR - i\left(\frac{i}{\omega C}\right) = iZ$$

Here, impedance is, $Z = R - j\left(\frac{1}{\omega C}\right)$



The modulus of impedance is, $|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$

and the potential difference lags the current by an angle,

$$\phi = \tan^{-1}\left|\frac{V_C}{V_R}\right| = \tan^{-1}\frac{X_C}{R} = \tan^{-1}\left(\frac{1/\omega C}{R}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

Illustration : 8

A 50 W, 100 V lamp is to be connected to an ac mains of 200 V, 50 Hz. What capacitance is essential to be put in series with the lamp?

Solution

As resistance of the lamp $R = \frac{V_o^2}{W} = \frac{100^2}{50} = 200\,\Omega$ and the mxiumum current $I = \frac{V}{R} = \frac{100}{200} = \frac{1}{2}A$; so when the lamp is put in series with a capacitance and run at 200 V ac, from V = IZ we have,

$$Z = \frac{V}{I} = \frac{200}{(1/2)} = 400\,\Omega$$

Now as in case of C-R circuit,

$$Z = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}, \quad \text{i.e.} \quad R^2 + \left(\frac{1}{\omega C}\right)^2 = 160000$$

or $\quad \left(\frac{1}{\omega C}\right)^2 = 16 \times 10^{-4} - (200)^2 = 12 \times 10^4$

So $\quad \frac{1}{\omega C} = \sqrt{12} \times 10^2$ or $C = \frac{1}{100\pi \times \sqrt{12} \times 10^2}F$

i.e., $\quad C = \frac{100}{\pi\sqrt{12}}\mu F = 9.2\,\mu F$

Illustration : 9

An A. C. source of angular frequency ω is fed across a resistor R and a capacitor C in series. The current registered is i. If now the frequency of the source is changed to $\omega/3$ (but maintaining the same voltage), the current in the circuit is found to be halved. Calculate the ratio of reactance to resistance at the original frequency.

Solution

At angular frequency ω, the current in R-C circuit is given by

$$i_{rms} = \frac{\varepsilon_{rms}}{\sqrt{\left\{R^2 + \left(1/\omega^2C^2\right)\right\}}} \quad (1)$$

When frequency is changed to $\omega/3$, the current is halved. Thus

$$\frac{i_{rms}}{2} = \frac{\varepsilon_{rms}}{\sqrt{\left\{R^2 + 1/\left(\omega/3\right)^2 C^2\right\}}} = \frac{\varepsilon_{rms}}{\sqrt{\left\{R^2 + \left(9/\omega^2C^2\right)\right\}}} \quad (2)$$

From equation (1) and (2), we have

$$\frac{1}{\sqrt{\left\{R^2 + \left(1/\omega^2C^2\right)\right\}}} = \frac{2}{\sqrt{\left\{R^2 + \left(9/\omega^2C^2\right)\right\}}}$$

Solving this equation, we get $3R^2 = \frac{5}{\omega^2C^2}$

Hence, the ratio of reactance to resistance is $\frac{(1/\omega C)}{R} = \sqrt{\left(\frac{3}{5}\right)}$

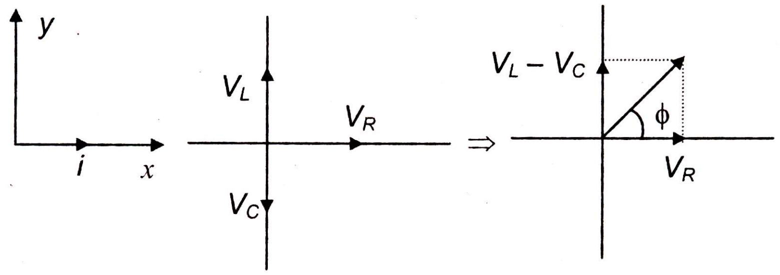
Series L-C-R Circuit

Now consider an ac circuit consisting of a resistor of resistance R, a capacitor of capacitance C and an inductor of inductance L are in series with an ac source generator.

Suppose in a phasor diagram current is taken along positive x-direction. Then V_R is along positive x-direction, V_L along positive y-direction and V_C along negative y-direction, as potential difference across an inductor leads the current by 90° in phase while that across a capacitor, lags by 90°.

;

$$V = \sqrt{V_R^2 + \left(V_L - V_C\right)^2}$$



So, we can write, $V = V_R + jV_L - jV_C = iR + j(iX_L) - j(iX_C) = iR + j\left[i\left(X_L - X_C\right)\right] = iZ$

Here, impedance is, $Z = R + j\left(X_L - X_C\right) = R + j\left(\omega L - \frac{1}{\omega C}\right)$

The modulus of impedance is, $|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$

and the potential difference leads the current by an angle,

$$\phi = \tan^{-1}\left|\frac{V_L - V_C}{V_R}\right| = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$$

$$\phi = \tan^{-1}\left(\frac{\omega L - \dfrac{1}{\omega C}}{R}\right)$$

RESONANCE IN SERIES L-C-R CIRCUIT

In an AC, $E = E_o \sin\omega t$ is applied to a circuit containing L, C and R in series as shown in figure, then as,

$$X = X_L - X_C = \left(\omega L - \frac{1}{\omega C}\right)$$

$$Z = \sqrt{R^2 + X^2} = \sqrt{R^2 + \left[\omega L - (1/\omega C)\right]^2}$$

So, $$I_o = \frac{E_o}{Z} = \frac{E_o}{\sqrt{R^2 + \left[\omega L - (1/\omega C)\right]^2}} \qquad (1)$$

and $$\phi = \tan^{-1}\frac{X}{R} = \tan^{-1}\left[\frac{\omega L - (1/\omega C)}{R}\right] \qquad (2)$$

So current in the circuit at any time t will be given by,

$$I = I_o \sin(\omega t - \phi)$$

Where I_o and ϕ are given by equations (1) and (2) resepctively.

From this it is clear that in case of series L-C-R circuit:

1. Current in the circuit may lag, lead or be in phase with the applied voltage depending on the fact that $X_L > X_C$, $X_L < X_C$ or $X_L = X_C$ respectively.

2. If $X_L = X_C$, i.e., $\omega L = \dfrac{1}{\omega C}$

   i.e. $\omega = \dfrac{1}{\sqrt{LC}}$ or $f = \dfrac{1}{2\pi\sqrt{LC}} = f_o$

   i.e., the frequency f of applied ac is equal to the natural frequency of the circuit $f_o = \dfrac{1}{2\pi\sqrt{LC}}$, the circuit is said to be in resonance.

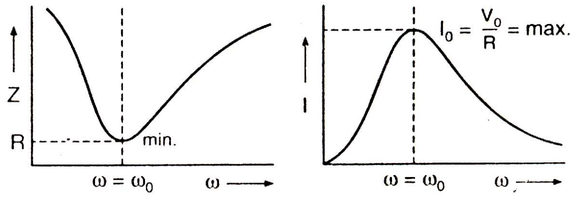
3. In case of series resonance, i.e. $f = f_o$:

   (i) $X_L = X_C$, i.e. inductive reactance is equal to capcitive reactance.

   (ii) $X = X_L - X_C = 0$, i.e. 'reactance' of the circuit is zero.

   (iii) $Z = \sqrt{R^2 + X^2} = R$, i.e. 'impedance' of the circuit is minimum and is equal to resistance.

   (iv) $I = \dfrac{V_o}{Z} = \dfrac{V_o}{R} = \text{max}.$, i.e., current in the circuit is maximum.



(A) (B)

(v) Before resonance, current in the circuit leads the applied voltage (as $X_L < X_C$) and after resonance it lags the applied voltage (as $X_C < X_L$) and at resonance $\phi = \tan^{-1}(X/R) = \tan^{-1}(0/R) = 0$, i.e., current is in phase with applied voltage.

(vi) Potential difference through L and C are same but 180° out of phase with respect to each other so that net PD across reactance is zero, i.e.,

$$V_X = V_L - V_C = 0 \quad \text{with} \quad V = V_R$$

(vii) The 'power factor' of the circuit.

$$PF = \cos\phi = \frac{R}{Z} = 1 = \text{max.} \quad (\text{as } Z = R = \text{min.})$$

and hence power consumed by the circuit,

$$P_{av} = V_{rms} I_{rms} \cos\phi = \frac{1}{2} V_o I_o = \text{max.}$$

(viii) The series resonant circuit is called 'acceptor circuit' as at resonance its impedance is minimum and it most readily accepts that current out of many currents whose frequency is equal to its natural frequency. In radio or TV tuning we receive the desired station by making the frequency of the circuit equal to that of the desired station.

(ix) At resonance as $I_o = E_o / R$,

So, $V_L = I_o X_L = \dfrac{\omega L}{R} E_o$

i.e., $V_L = QE_o$ with $Q = \dfrac{\omega L}{R}$

and is called 'quality factor' of the circuit. Thus, at resonance the voltage drop across inductance (or capacitance) is Q times the applied voltage.

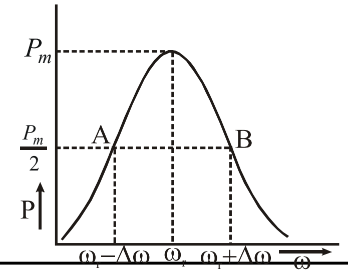
Hence, the chief characteristic of series resonant circuit is "voltage magnification".

Band Width And Q-Factor

Angular frequency variation with power in LCR series circuit.

$$P = P_m \sqrt{\frac{R^2}{\left[R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2\right]}}$$

Graph between P & ω as shown in the figure.



$$\omega_1 = +\frac{R}{2L} + \left(\omega_r^2 + \frac{R^2}{4L^2}\right)^{1/2} \text{ and } \omega_2 = -\frac{R}{2L} + \left(\omega_r^2 + \frac{R^2}{4L^2}\right)^{1/2}$$

Now, $\omega_1 - \omega_2 = \dfrac{R}{L}$ or $(\omega_r + \Delta\omega) - (\omega_r - \Delta\omega) = \dfrac{R}{L}$ or $2\Delta\omega = \dfrac{R}{L}$.

The frequency interval between half maximum power points is known as band width.

The ratio resonance frequency and band width is known as quality factor (Q).

$$\therefore \quad Q = \frac{\omega_r}{2\Delta\omega} = \frac{\omega_r L}{R}.$$

Q factor is a measure of the sharpness of resonance. Resonance will be sharp if the value of bandwidth $(2\Delta\omega)$ is small. This is of course possible only when the power-frequency curve fall steeply around $\omega = \omega_r$

Illustration : 10

A resistor of resistance R, an inductor of inductance L and a capacitor of capacitance C all are connected in series with an a.c. supply. The resistance of R is 16 ohm and for a given frequency, the inductive reactance of L is 24 ohm and capacitive reactance of C is 12 ohm. If the current in the circuit is 5 amp;, find

(a) the potential difference across R, L and C

(b) the impedance of the circuit,

(c) the voltage of a.c. supply,

(d) phase angle.

Solution

(a) Potential difference across resistance $V_R = iR = 5 \times 16 = 80 \text{ volt}$

Potential difference across inductance $V_L = i \times (\omega L) = 5 \times 24 = 120 \text{ volt}$

Potential difference across condenser $V_C = i \times (1/\omega C) = 5 \times 12 = 60 \text{ volt}$

(b) $Z = \sqrt{\left[R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2\right]} = \sqrt{\left[(16)^2 + (24-12)^2\right]} = 20 \text{ ohm}$

(c) The voltage of a.c. supply is given by

$V = iZ = 5 \times 20 = 100 \text{ volt}$

(d) Phase angle

$$\phi = \tan^{-1}\left[\frac{\omega L - (1/\omega C)}{R}\right] = \tan^{-1}\left[\frac{24-12}{16}\right] = \tan^{-1}(0.75) = 36°87'$$

Illustration: 11

A series circuit consists of a resistance of 15 ohm, an inductance of 0.08 henry and a condenser of capacity 30 micro farad. The applied voltage has a frequency of 500 radian/s. Does the current lead or lag the applied voltage and by what angle?

Solution

Here $\omega L = 500 \times 0.08 = 40 \text{ ohm}$

and $\dfrac{1}{\omega C} = \dfrac{1}{500 \times (30 \times 10^{-6})} = 66.7 \text{ ohm}$

$$\tan\phi = \frac{[\omega L - (1/\omega C)]}{R} = \frac{40 - 66.7}{15} = -1.78$$

$$\phi = -60.67°$$

Thus the current leads the applied voltage by 60.67°.

POWER IN AN AC CIRCUIT

In case of a steady current the rate of doing work is given by,

P = Vi

In an alternating circuit, current and voltage both vary with time, so the work done by the source in time interval dt is given by

dW = Vidt

Suppose in an ac, the current is leading the voltage by an angle ϕ. Then we can write,

$$V = V_o \sin\omega t \quad \text{and} \qquad i = i_o \sin(\omega t + \phi)$$

$$dW = V_o i_o \sin\omega t \sin(\omega t + \phi)\,dt = V_o i_o \left(\sin^2\omega t \cos\phi + \sin\omega t \cos\omega t \sin\phi\right)dt$$

The total work done in a complete cycle is

$$W = V_o i_o \cos\phi \int_0^T \sin^2\omega t\,dt + V_o i_o \sin\phi \int_0^T \sin\omega t \cos\omega t\,dt$$

$$= \frac{1}{2}V_o i_o \cos\phi \int_0^T (1 - \cos 2\omega t)\,dt + \frac{1}{2}V_o i_o \sin\phi \int_0^{\phi} \sin 2\omega t\,dt = \frac{1}{2}V_o i_o T \cos\phi$$

The average power dilivered by the source is, therefore,

$$P = \frac{W}{T} = \frac{1}{2}V_o i_o \cos\phi = \left(\frac{V_o}{\sqrt{2}}\right)\left(\frac{i_o}{\sqrt{2}}\right)(\cos\phi) = V\varepsilon_{rms} i_{rms} \cos\phi$$

or $\quad < P >_{\text{one cycle}} = V_{rms} i_{rms} \cos\phi$

Here, the term $\cos\phi$ is known as power factor.

It is said to be leading if current leads voltage, lagging if current lags voltage. Thus, a power factor of 0.5 lagging means current lags the voltage by 60° (as $\cos^{-1}0.5 = 60°$). The product of V_{rms} and i_{rms} gives the apparent power. While the true power is obtained by multiplying the apparent power by the power factor $\cos\phi$, Thus,

apparent power = $V_{rms} \times i_{rms}$

True power = apparent power $\times$ power factor.

For $\phi = 0^o$, the current and voltage are in phase. The power is thus, maximum ($V_{rms} \times i_{rms}$). For $\phi = 90^o$, the power is zero. The current is then stated as wattless. Such a case will arise when resistance in the circuit is zero. The circuit is purely inductive or capacitive.

The choke coil

In a direct current circuit, current is reduced by means of a rheostat (resistance). This results in a loss of electrical energy I^2R per sec as heat in the resistance.

The current in an ac circuit can however be reduced by means of a device which involves very smal amount of loss of energy. This device is called 'choke coil' or ballast and consists of a copper coil wound over a soft iron laminated core. This coil is put in series with the circuit in which current is to be reduced. As this circuit is a L-R circuit, the current in the circuit,

$$I = \frac{E}{Z} \quad \text{with} \quad Z = \sqrt{(R + r)^2 + (\omega L)^2}$$

So due to large inductance L of the coil, the current in the circuit is decreased appreciably. However, due to small resistance of the coil r, the power loss in the choke,

$$P_{av} = V_{rms} I_{rms} \cos\phi \to 0$$

As $\quad \cos\phi = \frac{r}{Z} = \frac{r}{\sqrt{r^2 + \omega^2 L^2}} \approx \frac{r}{\omega L} \to 0$

Illustration : 15

A series LCR with R = 20Ω , L = 1.5 H and C = $35\mu F$ is connected to a variable frequency 200 V a.c. supply. When the frequency of the supply equals the natural frequency of the circuit. What is the average power transferred to the circuit in one complete cycle?

Solution

When the frequency of the supply equals the natural frequency of the circuit, resonance occurs.

$\therefore$ Z = R = 20 ohm

$$i_{rms} = \frac{E_{rms}}{Z} = \frac{200}{20} = 10 \text{ A}$$

Average power transferred/cycle

$$P = E_{rms} i_{rms} \cos 0^\circ = 200 \times 10 \times 1 = 2000 \text{ watt}.$$

KEYS POINTS

- In an A.C. generator, mechanical energy is converted to electrical energy on the basis of electromagnetic induction. If an N turn coil of area A is rotated at f revolutions per second in a uniform magnetic field B, then the motional emf produced is sinusoidal.
  $\varepsilon = NBA(2\pi f)\sin(2\pi ft)$
  where we have assumed that that at time t = 0, the coil is perpendicular to the field.

- An alternating voltage $V = V_m \sin\omega t$ applied to a resistor R drives a current $I = I_m \sin\omega t$ in the resistor, $I_m = V_m / R$. The current is in phase with the applied voltage.

- When a value is given for ac voltage of current, it is ordinarily 240 V. This refers to the rms value of the voltage. The amplitude of this voltage is
  $V_m = \sqrt{2}\,V_{rms} = \sqrt{2}(240) = 340\,V$

- For an alternating current $I = I_m \sin\omega t$ passing through a resistor R, the average power loss $\overline{P}$ (averaged over a cycle) due to joule heating is $(1/2)I_m^2R.$ To express it in the same form as the DC power $(P = I^2R)$; a special value of current is used. It is called root mean square (rms) current and is denoted by I_{rms}.
  $$I_{rms} = \frac{I_m}{\sqrt{2}} = 0.707\,I_m$$
  Similarly, the rms voltage is defined by
  $V_{rms} = \frac{V_m}{\sqrt{2}} = 0.7070\,V_m$. We have $\overline{P} = I_{rms}V_{rms} = I_{rms}^2R$

- The power rating of an element used in ac circuits refers to its average powr rating.
- In an ac circuit, while adding voltages different elements, one should take care of their phases properly. For example, if V_R and V_C are voltages across R and C, respectively, in an RC circuit, then the total voltage across RC combination is $V_{RC} = \sqrt{V_R^2 + V_C^2}$ and not $V_R + V_C$ as $\frac{\pi}{2}$ out of phase of V_R.
- There are no power losses associated with pure capacitances and pure inductances in an ac circuit. The only element that dissipates energy in an ac circuit is the resistive element.
- The power factor in a RLC circuit is a measure of how close the circuit is to expending the maximum power.

- An ac voltage $V = V_m \sin\omega t$ applied to a capacitor drives a current in the capacitor $i = i_m \sin\left(\omega t + \frac{\pi}{2}\right)$.
  Here, $i_m = \frac{V_m}{X_C}$ and $X_C = \frac{1}{\omega C}$ is called capacitive reactance. The current through the capacitor is $\pi/2$ ahead of the applied voltage. As in the case of inductor, the average power supplied to a capacitor over one complete cycle is zero.
- For a series RLC circuit driven by voltage $V = V_m \sin(\omega t + \phi)$, the current is given by $I = I_m \sin\omega t$ where
  $I_m = \frac{V_m}{\sqrt{R^2 + (X_L - X_C)^2}}$ and $\phi = \tan^{-1}\frac{X_L - X_C}{R}$.

$Z = \sqrt{R^2 + \left(X_L - X_C\right)^2}$ is called the impedance of the circuit. The average power loss over a complete cycle is given by

$$\overline{P} = V_{rms} I_{rms} \cos\phi$$

The term $\cos\phi$ is called the power factor.

- Though in phasor diagram, voltage and current are represented by vectors, these quantities are not really vectors themselves. They are scalar quantities. It so happens that the amplitudes and phase of harmonically varying scalars combine mathematically in the same way as do the proejctions of rotating vectors of corresponding magnitudes and dierctions. The 'rotating vectors' that represent harmonically varying scalar quantities are introduced only to provide us with a simple way of adding these quantities using a rule that we already know as the law of vector addition.
- An interesting characteristic of a series RLC circuit is the phenomenon of resonance. The circuit exhibits resonance, i.e. the amplitude of the current is maximum at the resonant frequency, $\omega_o = \frac{1}{\sqrt{LC}}$. The quality factor Q defined by Q = $\frac{\omega_o L}{R} = \frac{1}{\omega_o CR}$ is an indicator of the sharpness of the resonance, the higher value of Q indicating sharper peak in the current.

$$\frac{d^2q}{dt^2} + \frac{1}{LC} q = 0$$

and therefore, the frequency ω of free oscillation is $\omega_o = \frac{1}{\sqrt{LC}}$. The energy in the system oscillates between the capacitor and the inductor but their sum or the total energy is constant in time.