1.Quadratic Equations & Inequations

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Points to Remember

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
4. $(a+b)^3 = a^3 + b^3 + 3ab(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$
5. $(a-b)^3 = a^3 - b^3 - 3ab(a-b) = a^3 - 3a^2b + 3ab^2 - b^3$
6. $a^2 - b^2 = (a+b)(a-b)$
7. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
8. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
9. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$
10. $a^3 + b^3 + c^3 = 3abc$, if $a+b+c=0.$

Quadratic Equations and Inequations

1. Sets and Intervals

A set is the collection of well-defined elements. For example, A is a set of class 5 students; B is the set of first five letters of English alphabets etc. But C, collection of all good students of class 5 is not a set, because we do not have a scale to measure the goodness of the students.

For example, A = {1, 2, 3, 4, 5} and B = {1, 2, 3 } are sets. Here, A is the set containing first five natural numbers and so its elements are written inside the opening and closing braces. These elements inside the braces are separated by commas.

Two Ways of Representing Sets

I. **Roster form**: In this method all the elements of the set are written between opening and closing curly braces, e.g. A = {1, 2, 3, 4, 5}.

II. **Set Builder Form**: In set builder form, any trend or property among the elements of the set is observed and this property is written inside the braces in the following way:

e.g. A = { x: x is a natural number, less than 6 }.

Here, all the elements of set B are also present in set B, so we say that set B is the subset of set A and we write it as $A \subset B$.

On the basis of number of elements, the sets are further divided into two categories; **finite** and **infinite** sets. The set A mentioned above is a finite set as it has countable or finite number of elements while the set of natural numbers, N = { 1, 2, 3, 4……..} is an example of infinite set.

Similarly, W, I or, Z, Q, R and C are infinite sets which is generally used to represent the set of whole numbers, set of integers, set of rational numbers, the set of real numbers and set of complex numbers respectively.

In the set of real numbers, the numbers which are not rational numbers are called **irrational numbers**, e.g., $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are rational numbers. The decimal presentation of irrational numbers are neither terminating nor repeating. The students please note that the numbers whose decimal presentation are either terminating or repeating are rational numbers. Such numbers can also be written in the form p/q, where p and q both belong to the set of integers (I) and also q has to be an integer which is not zero.

The order relation among these infinite sets is as under:

$$N \subset W \subset I \subset Q \subset R \subset C$$

Further, the set of irrational numbers is $Q' \subset R \subset C$. The students please note that the set of rational numbers Q and the set of irrational numbers Q' are disjoint sets as these two sets do not have any element common between them.

2. Basic Algebra

I. $(a+b)^2 = a^2 + b^2 + 2ab$

II. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

III. $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2$(sum of the products of a, b, c and d taken two at a time)

IV. $(a_1 + a_2 + a_3 + \ldots + a_n)^2 = \sum (a_1)^2 + 2\sum a_1 a_2$.

V. $1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$.

VI. $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2$

$$= \frac{n(n+1)(2n+1)}{6}.$$

VII. $(1+2+3+4+\ldots+n)^2 = \{\frac{n(n+1)}{2}\}^2$.

Illustration 1. Find the sum of the product of first n natural numbers taking two at a time.

Solution: $(1+2+3+4+\ldots+n)^2$

$$= 1^2+2^2+3^2+\ldots+n^2+2S_2.$$

$$\{\frac{n(n+1)}{2}\}^2 = \frac{n(n+1)(2n+1)}{6} + 2S_2$$

$$\therefore 2S_2 = \frac{n(n+1)}{2}\{\frac{n(n+1)}{2} - \frac{(2n+1)}{3}\}$$

$$\therefore S_2 = \frac{n(n+1)}{2}\{\frac{3n(n+1)-2(2n+1)}{6}\}$$

$$= \frac{n(n+1)}{4}\{\frac{3n^2-n-2}{6}\}$$

$$= \frac{n(n+1)}{4}\{\frac{(3n+2)(n-1)}{6}\}$$

$$\therefore S_2 = \frac{n(n+1)(n-1)(3n+2)}{24}.$$

3. Factorization Concept

I $a^2 - b^2 = (a+b)(a-b)$

II $a^2 + b^2$ = cannot be factorized into real factors.

III $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

IV $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

V $a^4 + b^4$ = cannot be factorized into real factors.

Following the trend, we can write, $a^{2n+1} + b^{2n+1}$

$$= (a+b)(a^{2n} - a^{2n-1}b + a^{2n-2}b^2 + - --- -- + b^{2n})$$

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ---- + b^n)$

e. g. $a^6 - b^6$

$$= (a-b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$$

$$= (a-b)\frac{\left[a^5\left\{1-\left(\frac{b}{a}\right)^6\right\}\right]}{1-\frac{b}{a}}$$

$$= \frac{(a-b)((a^6-b^6))}{a-b}$$

$a^5 + b^5 = (a+b) \times (a^4 - a^3b + a^2b - ab^3 + b^4)$

4. Surds and Rationalization

A number of the form, $\sqrt[n]{a}$, i.e. nth root of a rational number a, is known as a surd. So, all the surds are irrational numbers but the vice versa is not true.

5. Introduction to Expressions and Polynomials

Consider, the **expression**, 2x+3. It has two terms so it is also called binomial expression and when we equate this to zero, i.e. 2x+3=0, it becomes an **equation**. Here, x is a variable and 2, 3 are constants. The index of the variable is one, so this is also called a first degree or linear equation in one variable.

Thus the general form of first degree equation in one variable is ax+b=0, where x is variable and a, b are **arbitrary constants**. Similarly, 3x – 5=0, is another example of first degree equations in one variable, where 3, -5 are called **fixed constants**. Generally, we use a, b, c etc. as the arbitrary constants to write the general equations or expressions.

The expressions, in which the index of the variable belongs to the set of whole number (W) is called a polynomial.

In the expression 2x+3, the maximum index of the variable is 1 and it has two terms, namely; 2x and 3, so it is a polynomial. The polynomial containing two terms has special name- binomial.

Similarly, $2x^2$ is called monomial. $2x^2+3x+5$ is trinomial etc.

Order or Degree of a polynomial

The maximum index of the variable is called the **order** or, **degree** of the polynomial. The above considered polynomials; 2x+3, $2x^2$ and $2x^2+3x+5$ have orders, 1,2 and 2 respectively.

$a_ox^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots\ldots\ldots+a_n$, where a_o, a_1, a_2 .. are real numbers ($a_o \neq 0$), is an nth degree polynomial in one variable, x.

The expression, $2\sqrt{x}$ + 5 is not a polynomial, because the index of the variable is $\frac{1}{2} \notin$ W.

The real number 7 is a polynomial of degree 0, since 7= $7x^0$.

Illustration 2. Which of the following is not a polynomial?

(a) x^2+2x+3 (b) x^5+1 (c) $x^2 + \frac{1}{x}$ (d) 5.

Solution: In the first option, all the indices of the variable, x, e.g. 2, 1 and 0 belong to the set of whole numbers, so it is a polynomial.

In the second option, the co-efficients of first term, x^5 and the last term or constant term, x^0 is unity, while the co-efficients of x^4, x^3, x^2 and x are simultaneously zero (which belong to the set of whole numbers, W) as there is no appearance of such terms in the expression. So, the above expression is fulfilling all the conditions for being a polynomial.

The expression, $x^2 + \frac{1}{x}$ can be written as $x^2 + x^{-1}$. Here, the index of the second term is -1, which does not belong to W. So, this is not a polynomial.

In the last option, the expression is just a number, 5. As, 5 = 5 x^0, which fits well in the form of a valid polynomial. So, it is also a polynomial.

So, the correct option of the above question is , (c).

6. Linear Equations in Two Variables

The general form of the linear equation in two variable is **ax + by + c = 0**. Any first degree equation in two variables always represents a straight line in the x-y plane. The equations of type $x=k_1$ and $y=k_2$ are parallel to x and y-axes at distances of k_1 and k_2 respectively.

The unique equation of this category is 2x+3y+8=0. To find the unique values of both the variables, we must have one more different equation in x and y.

Let us consider the system of linear equations,

$$L_1 \equiv a_1x+b_1y+c_1=0 \quad \text{and}$$

$$L_2 \equiv a_2x+b_2y+c_2=0.$$

(i) The condition for the above system of equations to have unique solution is

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2}.$$

Here, we don't need to find the ratio $\frac{c_1}{c_2}$ and check weather it is equal to the other ratios or not. The graphs of the lines L_1 and L_2 intersect to each other at a unique point and the co-ordinate of the point of intersection of the lines gives the unique solutions for x and y.

(ii) The condition to have infinitely many solutions is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

We get such situations only when the two equations are not actually different. The second one is just got by multiplying the first one by any constant. In this case, the graphs of the two lines are coincident or one on the other. Hence, in other way we can say that these are intersecting at infinitely many points, giving the infinite pair of solutions for x and y.

(iii) The condition to have no solution is

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}.$$

The graphs of the lines become parallel or non-intersecting in this case, giving no solution for the variables. The system of equations is called **inconsistent** while in the previous cases the systems are called **consistent**.

7. Quadratic Equation in one variable

(a) Finding roots of general quadratic equation

The general form of the quadratic equation in one variable is

$$ax^2+bx+c=0, \text{ where, } a \neq 0.$$

Since, $a \neq 0$, therefore we can multiply both sides of the equation by 'a'.

$$a^2x^2 + abx + ac = 0 \times a.$$

$$\text{Or, } (ax)^2 + 2.ax.\frac{b}{2} + \left(\frac{b}{2}\right)^2 + ac - \left(\frac{b}{2}\right)^2 = 0.$$

$$\text{Or, } \left(ax+\frac{b}{2}\right)^2 = \frac{b^2}{4} - ac$$

(Note that the above process is called the **completing the square**, which we mostly use to find the range of the quadratic expressions.)

$$\text{Or, } \quad ax+\frac{b}{2} = \pm\frac{\sqrt{b^2-4ac}}{2}$$

(If $x^2 = 36 \Rightarrow x = \pm\sqrt{36}$ or, $x = \pm 6$)

$$\text{Or, } ax = -\frac{b}{2} \pm \frac{\sqrt{b^2-4ac}}{2}$$

$$\text{Or, } x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

Here, x has the values, $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$. (say, α and β)

α and β are also called the roots or zeros of the quadratic equation.

Sum of the roots,

$$\alpha + \beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) + \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}.$$

$$= -\frac{b}{a}$$

And product of the roots,

$$\alpha \times \beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) \times \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(\frac{b - \sqrt{b^2 - 4ac}}{2a}\right) \times \left(\frac{b + \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{c}{a}.$$

In the equation, $ax^2+bx+c=0$, since $a \neq 0$, therefore, dividing both sides by a, we get,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Or, **x^2-(sum of the roots)x + product of the roots=0**

Illustration 3. Without solving the equation $3x^2-7x + 2 = 0$, find the sum and product of roots?

Solution:

Here, sum of the roots = $\frac{-b}{a} = \frac{-(-7)}{3} = \frac{7}{3}$

and the product of roots = $\frac{c}{a} = \frac{2}{3}$.

Illustration 4. Form the equation whose roots are 2, 3 ?

Solution: Using the formula,

x^2- (sum of the roots) x + product of the roots = 0

or, $x^2 - (2+3)x + 2 \times 3 = 0$

or, $x^2 - 5x + 6 = 0$.

Illustration 5. Find the range of values of the expression $x^2 + x + 1$.

Solution: Here, x^2+x+1

$$= x^2 + 2.x.\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$

$$= \left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1$$

$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}.$$

[Here, the quantity $\left(x + \frac{1}{2}\right)^2 \geq 0$ as it is the square of an expression]

So, the minimum value of the expression is $\frac{3}{4}$.

(b) Nature of roots

The quantity **b^2 - 4ac** is called the discriminant, D of the quadratic equation.

The nature of the roots will be as follows:

I. $D>0 \Leftrightarrow$ roots will be real and unequal, provided a, b, c are real.

II. $D=0 \Leftrightarrow$ roots are real and equal, provided a, b, c are real.

III. $D<0 \Leftrightarrow$ roots are non-real conjugate complex, provided a, b, c are real.

IV. If D is a perfect square, the roots are rational, provided a, b, c are rational.

V. If D is not a perfect square, the roots are conjugate irrational, provided a, b, c are rational and $b \neq 0$.

Important:

- If a + b + c = 0, or sum of the co-efficients in any quadratic equation is zero, then 1 is a root of the equation $ax^2+bx+c=0$, which is real. Consequently, the other root will also be real.
- Irrational as well as complex roots occur in conjugate pairs provided their co-efficients in the corresponding equations are rationals and reals respectively.

Illustration 6. Find the roots of the equation $(a-b)x^2 + (b-c)x + (c-a)=0$.

Solution: An observation on the co-efficients says that their sum, i.e. (a-b) + (b-c) + (c-a) = 0. So, we can say that one of the roots of the above equation is 1.

Now, let the other root be α .

Then $1 \times \alpha = \frac{c-a}{a-b}$

$\therefore \alpha = \frac{c-a}{a-b}$

Illustration 7. If the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ are equal then prove that, $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$, i.e., a, b and c are in HP.

Solution: Here, also sum of the co-efficients is zero. So, 1 is root of the equation.

Since both roots are equal, therefore the other will also be one. Then $1 \times 1 = \frac{c(a-b)}{a(b-c)}$

$\therefore$ a(b-c)=c(a-b)

Or, ab-ac = ca – cb

Or, ab +cb = 2ac

Dividing both sides by abc, we get,

$\frac{2}{b} = \frac{1}{a} + \frac{1}{b}$. Hence, a, b and c are in H.P.

Illustration 8. Prove that roots of the equation $(b+c-2a)x^2+(c+a-2b)x+(a+b-2c)=0$ are rational, a, b and c are rationals.

Solution: Here, sum of the co-efficients is also zero. So, 1 is a root of the equation which is rational. So, other root will also be rational as a, b and c are rationals.

8. Common Roots

If α is common root in $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$,

$\Rightarrow a_1\alpha^2 + b_1\alpha + c_1 = 0$, and

$a_2\alpha^2 + b_2\alpha + c_2 = 0$,

$$\Rightarrow \frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - a_1c_2} = \frac{1}{a_1b_2 - a_2b_1}$$

(a) Symmetric Functions of roots

Any expression $f(\alpha,\beta)$ in α and β is said to be symmetric if it remains unchanged when α and β interchanged i.e., $f(\alpha,\beta) = f(\beta,\alpha)$.

Some of the symmetric functions of α and β are:

$$\alpha^2 + \beta^2;\ \alpha\beta;\ \alpha^3 + \beta^3;\ \alpha^2\beta^2;\ \alpha^2 + \beta^2 + \alpha\beta;$$

$$\frac{\alpha^3+\beta^3}{\alpha\beta};\ \frac{1}{\alpha}+\frac{1}{\beta};\ \frac{1}{\alpha^2}+\frac{1}{\beta^2};\ \alpha^2\beta + \alpha\beta^2.$$

$f(\alpha,\beta) = \alpha^2 - \beta$ is not symmetric function of α and β because $f(\alpha,\beta) \neq f(\beta,\alpha)$.

All symmetric functions of α and β can be expressed in terms of two symmetric functions

$\alpha + \beta$ and $\alpha\beta$.

Illustration 9. If α and β are the roots of the equation $2x^2+3x+7=0$, then find (I). $\alpha^2+\beta^2$ (ii) $\alpha^3+\beta^3$ (iii) $\frac{1}{\alpha}+\frac{1}{\beta}$ (iv) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$.

(b) Equations reducible to Quadratic form

There are certain equations which do not look in the form of quadratic equations but can be given quadratic form by some assumptions.

Solve the following equations:

1. $\sqrt{x}$ = x-2.
2. $x^4 + x^2 - 12 = 0$.

9. Identity

$f(x)=\phi(x)$, is an identity in x if f(x) and $\phi(x)$ have same values $\forall$ $x \in$ R.

An equation with arbitrary coefficients will be an identity iff each arbitrary co-efficient is separately equal to zero. In other way, $ax^2 + bx + c = 0$, will be an identity (or can have more than two solutions) if coefficients of each power of x is separately zero, i.e. a=0, b=0, c=0.

Illustration 10. For what value of a, the equation $(a^2-a-2)x^2+(a^2-4)x + a^2-3a+2 = 0$, will have three solutions (more than two solutions)? Does there exist a value of x for which the above will become an identity in a?

10. Higher degree equations in one variable

10.1 Solving cubic or higher degree equations

Introduction:

The general process of finding the roots of cubic equations is beyond the CBSE or JEE syllabus. We find one root of any cubic equation by hit and trial and then divide the expression corresponding to the given equation by the linear expression made with this root.

1. Solve: $X^3 + X^2 + X = 84$.
2. Solve: $2X^3 +5X^2 -2X +3 = 0$.

Note: The similar processes are followed to solve the equations of higher orders. Biquadratic or the equations of degree four are sometimes also solved by making it the square of the quadratic expression.

10.2 Relation between roots and coefficients of higher degree equations

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots + a_n ,$$

where, a_0, a_1, a_2 .. are real numbers ($a_0 \neq 0$) is an nth degree polynomial in one real variable x.

It has n roots (equal or unequal) which are either real or unreal.

S_r= sum of the products of the roots taken r at a time

$$=(-1)^r \frac{\text{coefficient of the (highest} - r\text{th)power}}{\text{coefficient of highest power of x}} .$$

1. If α , β , γ and δ are the roots of $2x^4-3x^3+7x-1=0$ then find $\sum \alpha$, $\sum \alpha\beta$, $\sum \alpha\beta\gamma$ and $\sum \alpha\beta\gamma\delta$.

2. If the sum of the roots of equation $x^3-px^2+qx-r = 0$ is zero then prove that pq=r.

3. Find the condition that the roots of the equation $x^3-px^2+qx-r = 0$ may be in A.P.

4. If $x^2 + x + 1$ is a factor of ax^3+bx^2+cx+d then find the real roots of $ax^3+bx^2+cx+d = 0$.

5. The real numbers x_1, x_2, x_3 satisfying the equation, $x^3 - x^2 + \beta x + \gamma = 0$ are in A.P. Find the interval in which β and γ lies?

6. f α , β , γ are the roots of $2x^3+x^2-7 = 0$ then find the value of $\sum [(\alpha/(\beta)] + \beta/(\alpha))$.

11. Exponential Equations

a^x, is called simple power – in which 'a' is called the base and x is called the index of the power. For a>0, this power becomes always positive for any real value of x and in this condition it is given a special name – **exponential**.

Solve the following exponential equations:

1. $18^{8-4x} = (54\sqrt{2})^{3x-2}$.

2. $3^x.8^{\left(\frac{x}{x+2}\right)} = 6$.

12. Inequations

12.1 Linear Inequations in one variable

<u>Solve $\forall$ x ∈ R.</u>

1. 2x>3.
2. -2x<3.
3. 2x+3>1.

12.2 System of linear Inequations in one variable.

Solve the following system of inequations:

1. 1. 2x-7 > 5-x and 11-5x $\le$ 1.
2. 4x+3 $\ge$ 2x+7 and 3x-5 <-2.

12.3 Graphical solution of linear Inequations in two variables.

Solve the following inequations:

1. Solve 3x + 2y > 6.
2. Solve $y \ge -\frac{1}{2}x$.
3. Solve: 2x-3 $\ge$ 0.

12.4 Graphical solution of system of linear Inequations in two variables.

Solve the following system of inequations graphically:

1. 2x+y-3 $\ge$ 0 and x - 2y + 1 $\le$ 0.

Solve the following quadratic Inequations $\forall$ x ∈ R.

1. $x^2-4x+3<0$
2. $(x^2-4x+3)(x-2)^2<0$, $x \ne 2$.

13.1. Logarithm

13.1 Introduction to Logarithm

Logarithm (or log) is a mathematical tool which makes the calculations easier. It converts the multiplication into addition and division into subtraction.

13.2 Laws of Logarithm

$\log_a N = x$ ⇔ $a^x = N$.

This is called exponential form of the equation.

This is called logarithmic form of the equation.

Note: The above two forms are interconvertible to each other. The condition for $\log_a N = x$ to be defined are: N > 0, a > 0, $a \neq 1$. For a^x to be defined a>0 and $a \neq$ 1 and in this case, a^x is called exponential and it is always positive $\forall x \in \mathbf{R}$.

1. $\log_a a$ = 1.
2. $\log_a 1$ = 0.
3. $\log_a(m \times n) = \log_a m + \log_a n$.
4. $\log_a(m \div n) = \log_a m - \log_a n$.
5. $\log_a n^k = k.\log_a n$.
6. $\log_{[(a)^k]} n = \frac{1}{k}.\log_a n$.
7. $\log_n m = \frac{1}{\log_m n}$.
8. $\log_n m = \frac{\log_a m}{\log_a n} \Rightarrow \log_a m = \log_n m \times \log_a n$. (This is called **base change formula**).
9. n = $a^{\log_a n}$.[Any number 'n' can be written in the form of exponent of any other number 'a'].

Exercises

1. Write the following in the logarithmic form:
   a. 2^7=128.
   b. 10^{-1}=0.1
   c. $(0.5)^2$=0.25
2. Express each of the following in exponential form:

(a) $\log_5 25$ = 2.

(b) $\log_2\left(\frac{1}{4}\right) = -$ 2.

3. Evaluate:

   (a) $\log_2 \sqrt{32}$.

   (b) $\log_{11}\left[\left(\frac{121\sqrt{14641}}{\sqrt[3]{1331}}\right)\right]$.

4. Show that: $\log 360 = 3\log 2 + 2\log 3 + \log 5$.
5. Solve for x:

(a) $\log_6 216 = x$.

(b) $\log 2 + \log(x+2) - \log(3x-5) = \log 3$.

6. Write each of the following in the standard form:

   (a) 5.678 (b) 0.05678 (c) 0.000005

7. Write the following in decimal form:

   (a) 3.2×10^{-2} (b) 0.04×10^{4}.

8. Find the characteristics of (i) log(59273) (ii) log(0.00253).

9. Find : (a) log(1873) (b) log(82.29) (c) log(0.000438).

11. Find: (a) antilog(0.2001) (b) antilog($\bar{2}.2935$) (c) antilog(-1.2467)

12. Simplify: 3.62×1.296, using logarithm.

13.3 Logarithmic Equations

1. $\log_3(2x^2+6x-5) = 1$.
2. $\log_{(x-2)}(3x^2-x-1) = 0$.
3. $\log_2[(25)^{x+3}-1] = \log_2[4(5)^{x+3}+1]$.
4. $\log_{(2x+3)}(6x^2+23x+21) =$

   $4 - \log_{(3x+7)}(4x^2+12x+9)$.
5. $\log_{1/7}[\log_{1/5}(\sqrt{(x+5)}+\sqrt{x})] = 0$.
6. $3\log_x 4 + 2\log_{4x} 4 + 3\log_{16x} 4 = 0$.
7. $7^{\log x} = 98 - x^{\log 7}$.
8. $4^{\log_2 2x} = 36$.

13.4 Logarithmic Inequations

Note: If $\log_a x > \log_a y$, then x > y iff a>1, else x < y if 0 < a < 1.

Logarithmic Inequations:

1. $\log_3(2x^2+6x-5) > 1$.
2. $\log_{(x-2)}(3x^2-x-1) > 0$.
3. $\log_x 2 . \log_{2x} 2 \log_2 4x > 1$.
4. $\log_{[(x)^2-1]}(x+1) < 1$.
5. Solve: $(x^2+x+1)^x < 1$.

14. Quadratic Equations II

14.1 Location of roots

1. Condition for a number k to lie between the roots of a quadratic equation is

   af(k) < 0.

2. Condition for both k_1 and k_2 to lie between the roots of a quadratic equation is
   (i) $af(k_1) < 0$. (ii) $af(k_2) < 0$.

3. Condition for a number k to be less than roots

   (i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k < \frac{-b}{2a}$.

4. Condition for a number k to be more than roots

   (i) $D \geq 0$ (ii) $af(k) > 0$ (iii) $k > \frac{-b}{2a}$.

5. Condition for both roots to lie between k_1 and k_2

   (i) D > 0 (ii) $af(k_1) > 0$ (iii) $af(k_2) > 0$

   (iv) $k_1 < \frac{-b}{2a} < k_2$.

6. Condition for exactly one root to lie between k_1 and k_2

   (i) $f(k_1).f(k_2) < 0$.

7. Both the roots are positive

(i) $\frac{-b}{a} > 0$ (ii) $\frac{c}{a} > 0$ (iii) D $\geq$ 0.

8. Both roots are –ve

(i) $\frac{-b}{a} > 0$ (ii) $\frac{c}{a} > 0$ (iii) D $\geq$ 0.

9. Roots of f(x) = 0 are opposite in sign:

(i) $\frac{c}{a} < 0$.

Solved Examples

Illustration 1. If $a < b$, then solution of $x^2 + (a+b)x + ab < 0$ is given by

a) $x < b$ or $x < a$

b) $a < x < b$

c) $x < a$ or $x > b$

d) $-b < x < -a$.

Solution: (D), Here, $x^2 + (a+b)x + ab < 0$

$\Rightarrow (x+a)(x+b) < 0 \Rightarrow -b < x < -a$.

Illustration 2. If $a, b, c \in R$ and $(a+b+c)c < 0$, then the quadratic equation $p(x) = ax^2 + bx + c = 0$ has: (a) A negative root (b) Two real root (c) Two imaginary root (d) None of these.

Solution: (b) $p(x) = ax^2 + bx + c = 0$,

Now, $a + b + c = p$ (1) and c = p(0)

According to question,

$(a+b+c)c < 0 \Rightarrow p(1)p(0) < 0$

$\Rightarrow p(x) = 0$, has one root in (0, 1).

$p(x) = 0$, has two real roots because if coefficients and one root are real, then other root would also be real.

Exercise

1. For what values of m $\in$ R, both the roots of equations x^2 -6mx+ $9m^2$ -2m +2=0 exceed 3?

2. Find the value of 'a' for which the inequality (x-3a)(x-a-3)<0 is satisfied $\forall x \in$ [1,3].

3. The equation $x^2 + ax + b^2 = 0$ has two roots, each of which exceed c. Prove that $c^2+ac+b^2>0$.

4. If α, β are the roots of the equation $4x^2-16x + \lambda = 0$, $\lambda \in$ R, such that, $1< \alpha <2$ and $2< \beta <3$. Find the number of integral solution of?

5. Let a, b and c be real. If $ax^2 + bx + c = 0$ has two real roots α and β where $\alpha <-1$ and $\beta >1$ then show that $1+ \frac{c}{a} + |\frac{b}{a}| < 0$.

6. If α is a real roots of the equation $ax^2 + bx + c = 0$ and β is a real root of the equation $-ax^2+ bx + c = 0$, then show that there is a root γ of the equation $\frac{a}{2}x^2 + bx + c = 0$ which lies between α and β .

14.2 More Problems on quadratic Equation

1. If the ratio of the roots of $lx^2+ nx + n = 0$, is p:q then prove that $\frac{\sqrt{p}}{q} + \frac{\sqrt{q}}{p} + \frac{\sqrt{n}}{l} = 0$.

2. A car travels 25 kms an hour faster than a bus for a journey of 500kms. The bus takes 10 hours more than the car. Find the speed of the car and the bus?

3. In writing a quadratic equation of the form $x^2+bx+c=0$, a student writes the coefficient of x incorrectly and finds the roots as 7, 8. Another student writes the constant term incorrectly and finds the roots as 8, -3. Find the correct equation?

15. Some Important Points

15.1 Second degree quadratic equation in two variables

The general form of the quadratic equation in two variables is, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. The equation is symmetrical in two variables x and y, i.e. if we treat y as constant then it becomes a quadratic equation in x and when we treat x as constant then it becomes a quadratic equation in y. Also, it has 6 arbitrary constants, a, b, c, f, g and h.

The quadratic function $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is resolvable into linear rational factors if

$$Abc + 2fgh - af^2 - bg^2 - ch^2 = 0,$$

$$\text{i.e.} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

15.2 Transformation of Equations

Transformation 1. Transformation of an equation into another equation whose roots are the reciprocals of the roots of the given equation.

Let, $f(x) = a^0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots\ldots\ldots$
$+a_{n-1}x + a_n = 0$ .... (i)

be the given equation. Let x and y be respectively the roots of the given equation and that of the transformed equation. Then,

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}.$$

Putting, $x = \frac{1}{y}$ in (i), we get:

$$\frac{a_0}{y^n} + \frac{a^1}{y^{n-1}} + \frac{a_2}{y^{n-2}} + \ldots\ldots\ldots$$
$$+\frac{a_{n-1}}{y} + a_n = 0 \Rightarrow a_ny^n + a_{n-1}y^{n-1} + \ldots.$$
$$+a_1y + a_0 = 0$$

This is the equation.

Note: Thus, to obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by 1/x in the given equation.

Illustration 1. Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ be in H.P.

Solution: The equation whose roots are reciprocals of the given equation is given by

$$\frac{1}{x^3} - \frac{p}{x^2} + \frac{q}{x} - r = 0$$

or, $rx^3 - qx^2 + px - 1 = 0$ .... (i)

Since, the roots of the given equation are in H.P. so, the roots of this equation are in A.P.

Let its roots be a-d, a and a+d.

Then, $(a - d) + a + (a + d) = -\left(-\frac{q}{r}\right)$

$$\Rightarrow 3a = \frac{q}{r} \Rightarrow a = \frac{q}{3r}$$

Since, 'a' is a root of (i), so,

$ra^3 - qa^2 + pa - 1 = 0$ $\Rightarrow r\left(\frac{q}{3r}\right)^3 - q\left(\frac{q}{3r}\right.$

$$\Rightarrow \frac{q^3}{27r^2} - \frac{q^3}{9r^2} + \frac{pq}{3r} - 1 = 0 \Rightarrow q^3 - 3q^3 + 9pqr$$

$$\Rightarrow 27r^2 - 9pqr + 2q^3 = 0.$$

Transformation 2. Transformation of an equation to another equation whose

roots are negative of the roots of a given equation

Let the given equation be $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ...$ $+a_{n-1}x + a_n = 0$

Note: Let x be a root of the given equation and y be a root of the transformed equation.

Then, y =-1 or, x=-y.

Thus, the transformed equation is obtained by putting x=-y in f (x)=0 and is therefore f(-y)=0

or, $a_0y^n - a_1y^{n-2} + ...+(-1)^2a_n = 0$

Illustration 2. The equation whose roots are negative of the roots of the equation: $x^3 - 5x^3 - 7x - 3 = 0$.

Solution: $(-x)^3 - 5(-x)^2 - 7(-x) - 3 = 0$

Or, $-x^3 - 5x^2 + 7x - 3 = 0$

or, $x^3 + 5x^2 - 7x + 3 = 0$.

Transformation 3. Transformation of equation to another equation whose roots are square of the roots of a given equation

Let x be a root of the given equation and y be that of the transformer equation. Thus,

$y = x^2 \Rightarrow x = \sqrt[2]{y}.$

Note: Thus, an equation whose roots are squares of the roots of a given equation is obtained by replacing x by $\sqrt{x}$ in the given equation.

Illustration 3. Form an equation whose roots are squares of the roots of the equation: $x^3 - 6x^2 + 11x - 6 = 0$.

Solution: Replacing x by $\sqrt{x}$ in the given equation, we get:

$$(\sqrt{x})^3 - 6(\sqrt{x})^2 + 11\sqrt[2]{x} - 6 = 0$$

$$\Rightarrow x^{\frac{3}{2}} + 11\sqrt{x} = 6x + 6$$

$$\Rightarrow \sqrt{x}(x + 11) = 6(x + 1)$$

$$\Rightarrow x(x + 11)^2 = 36(x + 1)^2$$

$$\Rightarrow x^3 - 14x^2 + 49x - 36 = 0$$

Transformation 4. Transformation of an equation to another equation whose roots are cubes of the roots of a give equation

Let x be a root of the given equation and y be that transformed equation.

Thus, $y = x^3 \Rightarrow x = y^{\frac{1}{3}}$

Note: Thus, an equation whose roots are cubes of the roots of a given equation is obtained

by replacing x by $x^{\frac{1}{3}}$ in the given equation.

Illustration 4. From an equation whose roots are cubes of the roots of equation: $ax^3 + bx^2 + cx + d = 0.$

Solution: Replacing by $x^{\frac{1}{3}}$ in the given equation, we get

$$a\left(x^{\frac{1}{3}}\right)^3 + b\left(x^{\frac{1}{3}}\right)^2 + c\left(x^{\frac{1}{3}}\right) + d = 0 \Rightarrow ax + d = -\left(bx^{\frac{2}{3}} + c\right.$$

$$\Rightarrow (ax + d)^3 = -\left(bx^{\frac{2}{3}} + cx^{\frac{1}{3}}\right)^3$$

$$\Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 = -\{b^3x^2 + c^3x + 3b$$

$$\Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^2 = -\{b^3x^2 + c^3x - 3b$$

$\Rightarrow \quad a^3x^3 + x^2(3a^2d - 3abc + b^3) + (3ad^2 - 3bcd + c$

This is the required equation.

15.3 Algebraic interpretation of Rolle's Theorem.

Note: You can leave this section at this stage. You can do it later after you study Differential Calculus.

Let f(x) be a polynomial having α and β as its roots such that $\alpha < \beta$. Then, $f(\alpha) = f(\beta) = 0$. Also, a polynomial function is everywhere continuous and differentiable. Thus, $f(x)$ satisfies all three conditions of Roll's theorem. Consequently, there exists $y \in (\alpha, \beta)$ such that $f'(y) = 0$ i.e., $f'(x) = 0$ at $x = y$.

In other words, $x = y$ is a root of $f'(x) = 0$. Thus, algebraically Roll's theorem can be interpreted as follows: Between any two roots of a polynomial f (x), there is a always a root of its derivative f(x).

Illustration 11. **If $a, b, c \in$ R such that $2a + 3b + 6c = 0$, show that the quadratic equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.**

Solution: Consider the polynomial

$f(x) = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx$, We have, $f(0) = 0$

And,

$$f(1) = \frac{a}{3} + \frac{b}{2} + c = \frac{2a + 3b + 6c}{6} = 0 [\because 2a + 3b + 6c = 0]$$

So, 0 and 1 are two roots of $f(x) = 0$. Therefore, $f'(x) = 0$ i.e. $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Illustration 12. If $a, b, c \in$ R and $a + b + c = 0$, then show that the quadratic equation $3ax^2 + 2bx + c = 0$ has at least one root in [-1,1].

Solution: Consider, $f(x) = ax^3 + bx^2 + cx$.

We have: $f(0) = 0$ and $f(1) = a + b + c = 0$

$\because a + b + c = 0$

So, 0 and 1 are root of $f(x) = 0$.

Therefore, $f'(x) = 0$, i.e., $3ax^2 + 2bx + c = 0$ has at least one real root between 0 and 1.

Also, (0, 1) $\in [-1,1]$. Hence $3ax^2 + 2bx + c = 0$ has at least one real root in [-1, 1].

15.4 Maximum number of positive and negative roots of a polynomial.

Descartes´rule of sign

Descartes´ rule of sign is used to determine the number of real zeros of a polynomial function.

It tells us that the number of positive real zeroes in a polynomial function f(x) is the same or less than by an even numbers as the number of changes in the sign of the coefficients. The number of negative real zeroes of the f(x) is the same as the number of changes in sign of the coefficients of the terms of f(-x) or less than this by an even number.

We will show how it works with an example.

Example 1: Determine the number of positive and negative real zeros for the given function (this example is also shown in our video lesson):

$$f(x) = x^5 + 4x^4 - 3x^2 + x - 6$$

Our function is arranged in descending powers of the variable, if it were not we would have to do that as a first step. Second we count the number of changes in sign for the coefficients of f(x). Here are the coefficients of our variable in f(x):

1 +4 -3 +1 -6

Our variables goes from positive(1) to positive(4) to negative(-3) to positive(1) to negative(-6).

Between the first two coefficients there are no change in signs but between our second and third we have our first change, then between our third and fourth we have our second change and between our 4^{th} and 5^{th} coefficients we have a third change of coefficients. Descartes´ rule of signs tells us that the we then have exactly 3 real positive zeros or less but an odd number of zeros. Hence our number of positive zeros must then be either 3, or 1.

In order to find the number of negative zeros we find f(-x) and count the number of changes in sign for the coefficients:

$$f(-x) = (-x)^5 + 4(-x)^4 - 3(-x)^2 + (-x) - 6 =$$
$$= -x^5 + 4x^4 - 3x^2 - x - 6$$

Here we can see that we have two changes of signs, hence we have two negative zeros or less but a even number of zeros.

Totally we have 3 or 1 positive zeros or 2 or 0 negative zeros.

Example 2: In the equation, $\mathbf{x^3 + 3x^2 + 7x - 11 = 0}$, the signs of coefficients are: + + + −

As there is just one change of sign, the number of positive roots of $\mathbf{x^3 + 3x^2 + 7x - 11 = 0}$ is at most 1.

16. Use of Calculus in Solving Quadratic Equations

Result 1. If f(λ) and f(μ) are of opposite signs then the equation f(x)=0 has a real root lying between the real numbers λ and μ .

Illustration 1. If f(x) = $x^3 - 4x^2 - 5x + 1$ then show that the equation,$x^3 - 4x^2 - 5x + 1 = 0$ has a root lying between 4 and 5.

Solution: f(x) = $x^3 - 4x^2 - 5x + 1$

f(4) = -19 < 0 and f(5) > 0.

∴ $x^3 - 4x^2 - 5x + 1 = 0$ has a real root lying between 4 and 5.

Result 2. If the equation f(x)=0 has two real roots α and β then f'(x)=0 will have real a root lying between α and β . (Rolle's Theorem)

Result 3. If α is a repeated root then, α is a root of the derived equation.

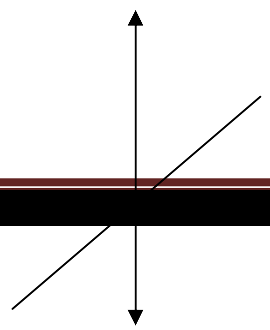
Result 4. If α is repeated root common in f(x)=0 and ϕ (x)=0 then α is common root in f'(x)=0 and ϕ '(x)=0.

Result 5. If (x- α)^2 is a factor f(x) then, (x- α) is a factor of f(x) as well as f'(x).

17. Graphs of Modulus functions

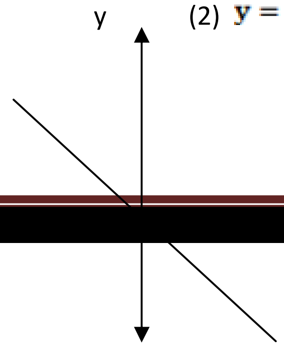
(1) $\mathbf{y = x}$

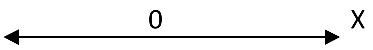
y

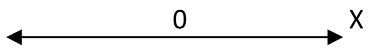


(2) $\mathbf{y = -x}$

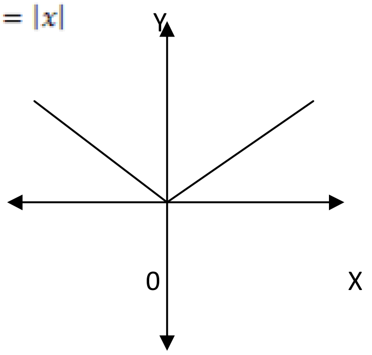
y



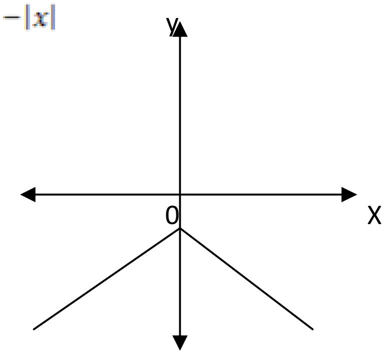




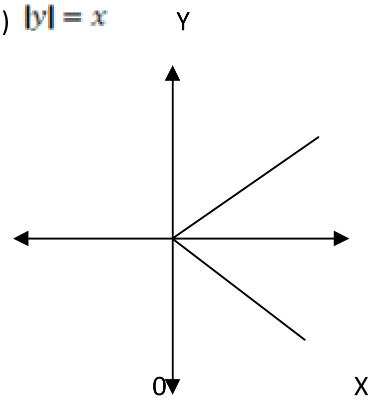
(3) $y = |x|$



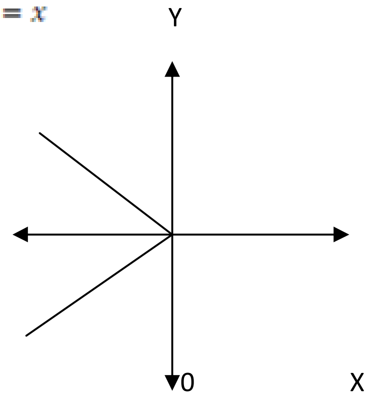
(4) $y = -|x|$



(5) $|y| = x$

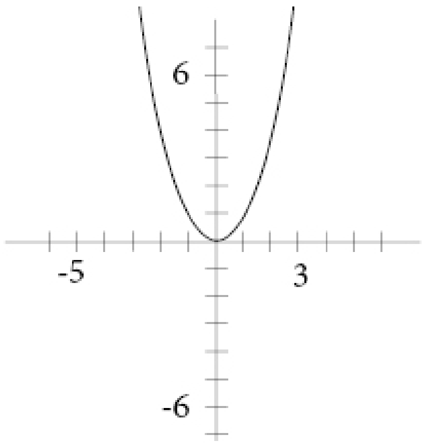


(6) $-|y| = x$

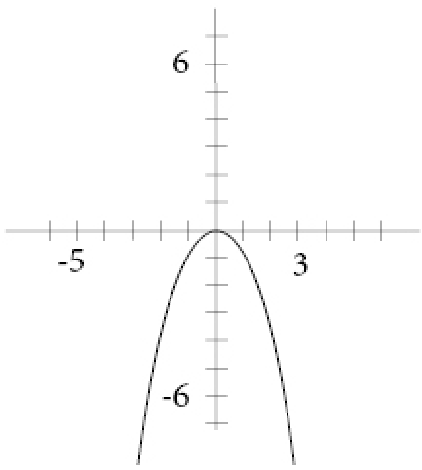


18.0 Graph of Quadratic Polynomials

We arrive at the following graph when we draw up a quadratic function such as y = x^2:



We can easily see that we are not dealing with a straight line but a parabola, thus it is referred to as a non-linear function. When one has a positive coefficient before x^2 we have a minimum value, and if we have a negative coefficient we have a maximum value instead. See the graph below where $y = -x^2$:



A rule of thumb reminds us that when we have a positive symbol before x^2 we get a happy expression on the graph **:)** and a negative symbol renders a sad expression **:(** .

We graph our quadratic function in the same way as we graph a linear function. First we make a table for our x- and y-values. From the x values we determine our y-values.

Last we graph our matching x- and y-values and draw our parabola.

Example: Graph $f(x)=(x-4)^2+1$.

The quadratic polynomial $ax^2 + bx + c$ is a function of x. So this can be expressed as

$$y = f(x) = ax^2 + bx + c$$

The graph of the function is a parabola whose concavity is upward for a>0 and downward for a<0. The intersection of the curve with the y-axis is found by substituting x=0 in the function, that is,

$$y = f(0) = a(0)^2 + b(0) + c = c$$

So , (0, c) is the point of intersection of the curve with y-axis.

The intersection of the graph with x-axis, depends on the discriminant, $D = b^2 - 4ac$. The following cases may arise:

Case I: when a>0 and D>0,

The concavity will be upward, cutting the x-axis at two distinct points.

Case II: when a<0 and D>0,

The concavity will be downward, cutting the x-axis at two distinct points.

Case III: when a>0 and D = 0,

The concavity will be upward, touching the x-axis at just one point.

Case IV: when a<0 and D = 0,

The concavity will be downward, touching the x-axis at point.

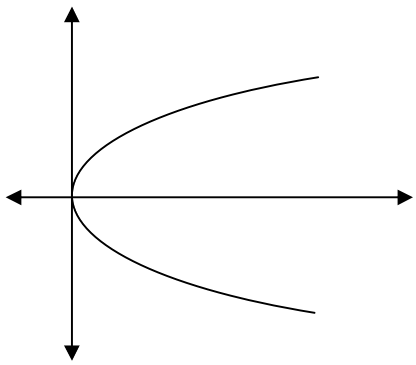
Case V: when a>0 and D<0,

The concavity will be upward, and the graph will be above the x-axis.

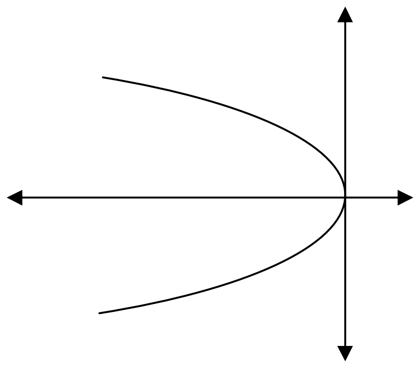
Case VI: when a<0 and D<0,

The concavity will be downward, and the graph will be below the x-axis.

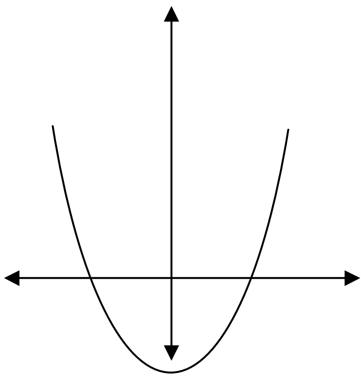
(1) $y^2 = 4ax$



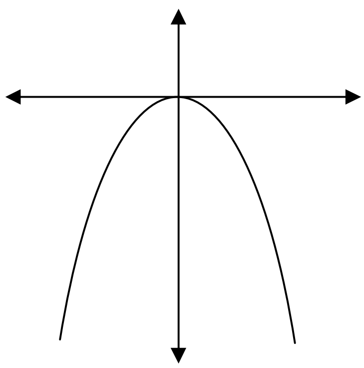
(2) $y^2 = -4ax$



(4)



$x^2 = 4ay$ (4) $x^2 = -4ay$



(5) $y^2 = ax^2$ Y

Greater is ‘a’ steeper is curve.

X

Illustration 3. The diagram show the graph of $y = ax^2 + bx + c$. Then,

(A) $a > 0$ (B) $b < 0$ (C) $c > 0$ (D) $b^2 - 4ac = 0$

Solution: (B) &(C). As it is clear from the figure that it is parabola opens downward i.e., $a < 0$.

$\Rightarrow$ It is $y = ax^2 + bx + c$ i.e. degree two polynomial

Now, if $ax + bx + c = 0$

$\Rightarrow$ it has two roots x_1 and x_2 as it cuts the axis at two distinct point x_1 and x_2.

Now from the figure it is also clear that $x_1 + x_2 < 0$. (i.e., sum of roots are negative)

$\Rightarrow \frac{-b}{a} < 0 \Rightarrow \frac{b}{a} > 0 \Rightarrow b < 0 \Rightarrow$ (B) is correct.

As the graph of $y = f(x)$ cuts the + y-axis at (0, c) where $c > 0 \Rightarrow$ (C) is correct.

Illustration 4. The adjoining figure shows the graph of $y = ax^2 + bx + c$. Then,

(A) $a > 0$ (B) $b^2 < 4ac$ (C) $c > 0$ (D) a and b are of opposite signs.

Solution: (A) & (D). As it is clear from the figure that it is a parabola which opens downwards i.e. $a < 0$.

$\Rightarrow$ (C) is correct.

$\Rightarrow$ It is $y = ax^2 + bx + c$ i.e. Degree two polynomial.

Now, if $ax^2 + bx + c = 0$

$\Rightarrow$ it has two roots x_1 and x_2 as it cuts the axis at two distinct points x_1 and x_2.

Now from the figure it is also clear that $x_1 + x_2 > 0$. (i.e. sum of roots are positive)

$\Rightarrow$ $\frac{-b}{a} > 0 \Rightarrow \frac{b}{a} > 0$ $\Rightarrow$ a and b are of opposite sings.

$\Rightarrow$ (D) is correct.

As D > 0 and $f(x) = c < 0$, (B) and (C) are wrong.

Previous Years NDA Questions

NDA 2018 Questions

1. What is the value of $\log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}}$ equal to ?

(a) $3\log_2 7$ (b) $1 - 3\log_2 7$ (c) $1 - 3\log_7 2$ (d) 7/8.

Solution : **(c)** Let x= $\log_7 \log_7 \sqrt{7\sqrt{7\sqrt{7}}}$

$\Rightarrow 7^x = \log_7 \sqrt{7\sqrt{7\sqrt{7}}}$

$\Rightarrow 7^{7^x} = \sqrt{7\sqrt{7\sqrt{7}}} = (7\sqrt{7\sqrt{7}})^{1/2} = \{7\,(7\sqrt{7})^{1/2}\}^{1/2} = (7.\ 7^{3/4})^{1/2} = 7^{7/8}$

$\Rightarrow 7^x = \frac{7}{8}$

$\Rightarrow x = 1 - \log_7 8 = 1 - 3\log_7 2$.

2. Consider the following expressions :

(1) $x + x^2 - \frac{1}{x^2}$ (2) $\sqrt{ax^2 + bx + x - c + \frac{d}{x} - \frac{e}{x^2}}$ (3) $3x^2 - 5x + ab$ (4) $\frac{2}{x^2 - ax + b^2}$ (5) $\frac{1}{x} - \frac{2}{x+5}$

Which of the above are rational expressions ?

(a) 1, 4 and 5 only. (b) 1, 3, 4 and 5 only (c) 2, 4 and 5 only (d) 1 and 2 only.

Solution : **(b)** Rational expressions are expressions which are the ratio of two polynomials.

3. A, B and C are subsets of universal set, then which one of the following is **not** correct ?

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (b) $A' \cup (A \cup B) = (B' \cap A)' \cup A$

(c) $A' \cup (B \cup C) = (C' \cap B)' \cap A'$ (d) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

Solution :(a) For the option (a) LHS$\neq$ RHS.

NDA 2019 Questions

(with Explanatory Answers)

1. If both p and q belongs to the set, {1, 2, 3, 4}, then how many equations of the form $px^2 + qx + 1 = 0$ will have real roots?

(a) 12 (b) 10 (c) 7 (d) 6

Solution : (c) For the equation $px^2 + qx + 1 = 0$, to have real roots, the discriminant, q^2-4p has to be grater than or equal to zero.

As both p and q belongs to the set {1, 2, 3, 4}, $q^2-4p \geq 0$ under the following conditions:

Case I. when q=4, p can take the values 4, or, 3, or, 2, or 1; in 4 ways- which will make the expression $q^2-4p \geq 0$.

Case II. when q=3, p can take the values only 2, or 1; in 2 ways- which will make the expression $q^2-4p \geq 0$.

Case III. when q=2, p can take the value only 1; in 1way- which will make the expression $q^2-4p \geq 0$.

Therefore, the total no. of ways is 4+2+1=7 ways.

2. If A, B, and C are the subsets of a given set, then which of the following relations is not correct?

(a) $A \cup (A \cap B) = A \cup B$ (b) $A \cap (A \cup B) = A$

(c) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ (d) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Solution: (a) $A \cup (A \cap B) = A$, which is not equal to $A \cup B$. So, option (a) is not correct.

3. What is the value of k for which the sum of the squares of the roots of the equation $2x^2 - 2(k-2)x - (k+1)=0$, is minimum ?

(a) -1 (b) 1 (c) 3/2 (d) 2.

Solution: **(c)** $2x^2 - 2(k-2)x - (k+1)=0$

Let the roots of the equation are a and b.

$a+b = k-2$ and $ab= -(k+1)/2$

Therefore, $a^2 + b^2 = (a+b)^2-2ab = (k-2)^2 +k+1 = k^2 -3k +5= k^2 -2.(3/2)k +(9/4)+2.75=(k-3/2)^2+2.75$

Therefore, $a^2 + b^2$ is minimum when k=3/2.

4. If $|x^2-3x+2|> x^2-3x+2$, then which of the following is correct?

(a) $x \le 1$ or $x \ge 2$ (b) $1 \le x \le 2$ (c) $1 < x < 2$ (d) x can take any real value except 3 and 4.

Solution: **(a)** $|x^2-3x+2|> x^2-3x+2$

For the above inequation to be defined, the quantity inside the modulus has to be greater than or equal to zero.

i.e., $x^2-3x+2 \ge 0$. Solving which we get, **$x \le 1$ or $x \ge 2$.**

5. If the roots of the equation $a(b - c)x^2 + b(c - a) x + c(a - b)= 0$ are equal then prove that, $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$, i.e., a, b and c are in HP. [NDA 2019]

Solution: Here, also sum of the co-efficients is zero. So, 1 is root of the equation. Since both roots are equal, therefore the other will also be one.

Then, $1 \times 1 = \frac{c(a-b)}{a(b-c)}$

$\therefore a(b-c)=c(a-b)$

$\Rightarrow ab-ac = ca - cb$

$\Rightarrow ab +cb = 2ac$

Dividing both sides by abc, we get, $\frac{2}{b} = \frac{1}{a} + \frac{1}{b}$. Hence, a, b and c are in H.P.

6. If a set A contains 3 elements and set B contains 6 elements, then what is the minimum number of elements that $A \cup B$ contains?

(a) 3 (b) 6 (c) 8 (d) 9

Solution : (b) The answer is obvious and that is 6 by the definition of union of two sets.

7. What are the values of x that satisfy the equation $\begin{vmatrix} x & 0 & 2 \\ 2x & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 3x & 0 & 2 \\ x^2 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix}$ **= 0 ?**

(a) $-2\pm\sqrt{3}$ **(b)** $-1\pm\sqrt{3}$ **(c)** $-1\pm\sqrt{6}$ **(d)** $-2\pm\sqrt{6}$.

Solution : Expanding along row, we get,

$$x + 2(2x-2) + 3x(1) + 2x^2 = 0$$

$$\Rightarrow 2x^2 + 8x-4 = 0$$

$$\Rightarrow x^2 + 4x-2 = 0$$

$$\therefore x = \frac{-4 \pm \sqrt{24}}{2} = -2\pm\sqrt{6}$$

8. If x + a + b + c = 0, then what is the value of $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix}$ **?**

(a) 0 **(b)** $(a + b + c)^2$ **(c)** $a^2 + b^2 + c^2$ **(d) a + b + c – 2.**

Solution: **(a)** (x+a){(x+b)(x+c)-bc} -b{ax+ac-ac} + c{ab-ax-ab}

$(x+a)(x^2 +xb+xc -bax - cax)$

=(x+a)x(x +b+c) –ax(b+c)

= (x+a)x(-a) –ax(-a-x)

$= -ax^2 - a^2x + a^2x + ax^2$

= 0

NDA 2020 Questions

1. If $1.5 \le x \le 4.5$, then which of the following is correct? **[NDA 2020]**
   (a) (2x-3)(2x-9) > 0 (b) (2x-3)(2x-9) < 0 (c) (2x-3)(2x-9) $\ge$ 0 (d) (2x-3)(2x-9) $\le$ 0.

Solution : (d) For the inequation, (2x-3)(2x-9)>0, the roots of the corresponding equation are 1.5 and 4.5. Thus, the inequation, (2x-3)(2x-9)>0 would be satisfied for the values outside the interval $1.5 \le x \le 4.5$. Therefore the option (a) is not true.
And, it obvious that (d) is the correct option.

2. What is the value of $\frac{1}{10}\log_5 1024 - \log_5 10 - \frac{1}{5}\log_5 3125$? **[NDA 2020]**

(a) 0 (b) 1 (c) -2 (d) 3.

Solution : **(c)** $\frac{1}{10}\log_5 1024 - \log_5 10 - \frac{1}{5}\log_5 3125$

$= \frac{1}{10}\log_5 2^{10} - \log_5(2\times 5) - \frac{1}{5}\log_5 5^5$

$= \frac{1}{10} 10\times\log_5 2 \; \log_5(2) - \log_5(5) - \frac{1}{5} 5.\log_5 5$

$= \log_5 2 - \log_5(2) - 1 - 1$ = -2.

3. If x=$\log_c ab$, y=$\log_a bc$ and z=$\log_b ac$, then which of the following is correct? **[NDA 2020]**

(a) xyz=1

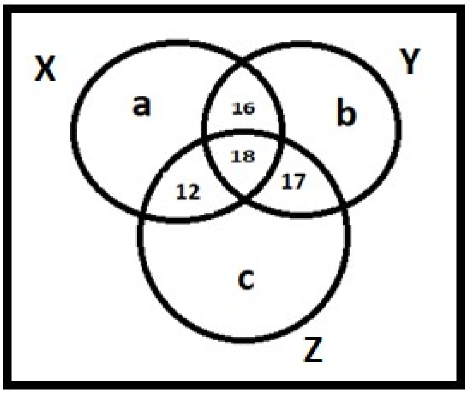
(b) x+y+z=1

(c) $(1+x)^{-1} + (1+y)^{-1} + (1+z)^{-1} = 1$

(d) $(1+x)^{-2} + (1+y)^{-2} + (1+z)^{-2} = 1$

Solution : x=$\log_c ab$, y=$\log_a bc$ and z=$\log_b ac$

Consider the following Venn diagram, where X, Y and Z are three sets. Let the elements in Z be denoted by n(Z) which is equal to 90, then answer the next 3 questions.



4. If the number in elements in Y and Z are in the ratio 4:5, then what is the value of b?

(a) 18 (b) 19 (c) 21 (d) 23. **[NDA 2020]**

Solution : **(c)** As n(z)=90, so n(Y)=72. [Since the ratio of elements in X and Y is 4:5]
Therefore, the no. of elements in b=72-(16+18+17)=72-51=21.

5. What is the value of $n(X) + n(Y) + n(Z) - n(X \cap Y) - n(Y \cap Z) - n(X \cap Z) + n(X \cap Y \cap Z)$?

(a) a + b + 43 (b) a + b + 63 (c) a + b + 96 (d) a + b + 106. **[NDA 2020]**

Solution : **(d)** n (X) + n (Y) + n (Z) – n (X $\cap$ Y) – n (Y $\cap$ Z) – n (X $\cap$ Z) + n (X $\cap$ Y $\cap$ Z) = n(X $\cup$ Y $\cup$ Z)
= a+b+c+12+16+18+17= a+b+43+12+16+18+17= a+b+106.

6. If the number of elements belonging to neither X, nor Y, nor Z is equal to p, then what is the number of elements in the complement of X? **[NDA 2020]**

(a) p + b + 60 (b) p + b + 40 (c) p + a + 60 (d) p + a + 40.

Solution : (a) Complement of X = X' = p + b + 17 + c = p + b + 60.