- **DEFINITION OF A SEQUENCE**

A succession of numbers $a_1, a_2, a_3..., a_n$, ... formed, according to some definite rule, is called a sequence.

- **ARITHMETIC PROGRESSION (A.P.)**

A sequence of numbers $\{a_n\}$ is called an arithmetic progression, if there is a number d, such that $d = a_n - a_{n-1}$ for all n. d is called the common difference (C.D.) of the A.P.

(i) Useful Formulae

If a = first term, d = common difference and n is the number of terms, then

(a) nth term is denoted by t_n and is given by

$t_n = a + (n - 1)d$.

(b) Sum of first n terms is denoted by S_n and is given by

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

or $S_n = \frac{n}{2}(a + l)$, where l = last term in the series i.e., $l = t_n = a + (n - 1)d$.

(c) If terms are given in A.P., and their sum is known, then the terms must be picked up in following way

- For three terms in A.P., we choose them as $(a - d), a, (a + d)$
- For four terms in A.P. , we choose them as $(a - 3d), (a - d), (a + d), (a + 3d)$
- For five terms in A.P., we choose them as $(a - 2d), (a - d), a, (a + d), (a + 2d)$ etc.

(ii) Useful Properties

- If $t_n = an + b$, then the series so formed is an A.P.
- If $S_n = an^2 + bn$ then series so formed is an A.P.
- If every term of an A.P. is increased or decreased by some quantity, the resulting terms will also be in A.P.
- If every term of an A.P. is multiplied or divided by some non-zero quantity, the resulting terms will also be in A.P.
- In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to sum of first and last terms.
- Sum and difference of corresponding terms of two A.P.'s will form an A.P.
- If terms $a_1, a_2, ..., a_n, a_{n+1}, ..., a_{2n+1}$ are in A.P., then sum of these terms will be equal to $(2n + 1)a_{n+1}$.
- If terms $a_1, a_2, ..., a_{2n-1}, a_{2n}$ are in A.P. The sum of these terms will be equal to $(2n)\left(\frac{a_n + a_{n+1}}{2}\right)$.

Illustration 1:

The m^{th} term of an A.P. is n and its nth term is m. Prove that its pth term is m + n – p. Also show that its (m + n) th term is zero.

Solution:

Given $T_m = a + (m - 1)\,d = n$ and $T_n = a + (n - 1)\,d = m$

Solving we get, $d = -1$ and $a = m + n - 1$

$\therefore \quad T_p = a + (p - 1)d = m + n - 1 + (p - 1)(-1) = m + n - p$

Now, $T_{m+n} = a + (m + n - 1)d = (m + n - 1) + (m + n - 1)(-1) = 0$.

Illustration 2:

Find the number of terms in the series 20, $19\frac{1}{3}$, $18\frac{2}{3}$, ... of which the sum is 300. Explain the double answer.

Solution:

Clearly here a = 20, $d = -\frac{2}{3}$ and $S_n = 300$.

$\therefore \quad \left(\frac{n}{2}\right)\left(2\times 20 + (n-1)\left(-\frac{2}{3}\right)\right) = 300$. Simplifying, $n^2 - 61n + 900 = 0 \Rightarrow n = 25$ or 36.

Since common difference is negative and $S_{25} = S_{36} = 300$, it shows that the sum of the eleven terms i.e., $T_{26}, T_{27}, \ldots, T_{36}$ is zero.

Illustration 3:

In an A. P., if the p^{th} term is $\frac{1}{q}$ and the q^{th} term is $\frac{1}{p}$, prove that the sum of the first pq terms must be $\frac{1}{2}(pq + 1)$.

Solution:

$$T_p = a + (p-1)d = \frac{1}{q}$$

$$\& T_q = a + (q-1)d = \frac{1}{p}$$

Solving T_p & T_q, are get

$$a = d = \frac{1}{pq}$$

$$S_{pq} = \frac{pq}{2}\left[2a + (pq-1)d\right] = \frac{pq+1}{2}$$

Illustration 4:

If the sum of n terms of an A. P. is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution:

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$= n\left(a - \frac{d}{2}\right) + n^2\frac{d}{2}$$

on comparing S_n with given sum

$$a - \frac{d}{2} = p \text{ and } q = \frac{d}{2}$$

$$\Rightarrow a = p + q \ \& \ d = 2q$$

Illustration 5:

If $a + b + c \neq 0$ and $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A. P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A. P.

Solution:

$\frac{b+c}{a}+1, \frac{c+a}{b}+1, \frac{a+b}{c}+1$ also in A.P.

$\Rightarrow \frac{a+b+c}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

- **GEOMETRIC PROGRESSION (G.P.)**

A sequence of the numbers $\{a_n\}$, in which $a_1 \neq 0$, is called a geometric progression, if there is a number $r \neq 0$ such that $\frac{a_n}{a_{n-1}} = r$ for all n then r is called the common ratio (C.R.) of the G.P.

(i) Useful Formulae

If a = first term, r = common ratio and n is the number of terms, then

(a) n^{th} term, denoted by t_n, is given by $t_n = ar^{n-1}$

(b) Sum of first n terms denoted by S_n is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}$$

In case r = 1, S_n = na.

(c) Sum of infinite terms (S_∞)

$$S_\infty = \frac{a}{1-r} \text{ (for } |r| < 1 \,\&\, r \neq 0)$$

(d) If terms are given in G.P. and their product is known, then the terms must be picked up in the following way.

- For three terms in G.P., we choose them as $\frac{a}{r}$, a, ar
- For four terms in G.P., we choose them as $\frac{a}{r^3}, \frac{a}{r}$, ar, ar^3
- For five terms in G.P., we choose them as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$ etc.

(ii) Useful Properties

(a) The product of the terms equidistant from the beginning and end is constant, and it is equal to the product of the first and the last term.

(b) If every term of a G.P. is multiplied or divided by the some non-zero quantity, the resulting progression is a G.P.

(c) If a_1, a_2, a_3 ... and b_1, b_2, b_3, ... be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2 ... and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}$ ... will also form a G.P. Common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

(d) If $a_1, a_2, a_3, \ldots$ be a G.P. of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$ will be an A.P. and conversely.

Illustration 6:

The first term of an infinite G..P is 1 and any term is equal to the sum of all the succeeding terms. find the series.

Solution:

Given that $T_p = (T_{p+1} + T_{p+2} + \ldots.. \infty)$ or, $ar^{p-1} = ar^p + ar^{p+1} + ar^{p+2} + \ldots$

$$\therefore\ r^{p-1} = \frac{r^p}{1-r} \text{ [sum of an infinite G.P.]}$$

$$\therefore\ 1 - r = r \Rightarrow r = \frac{1}{2}\text{. Hence the series is } 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\infty.$$

Illustration 7:

If the first and the n^{th} terms of a G. P. are a and b, respectively, and if P is the product of first n terms, prove that $P^2 = (ab)^n$.

Solution:

$b = ar^{n-1}$ .... (i)

$$p = (a)(ar)(ar^2)\ldots\ldots\ldots\ldots(ar^{n-1})$$

$$= a^n r^{1+2+\ldots\ldots\ldots\ldots+n-1} = a^n r^{\frac{n(n-1)}{2}} \Rightarrow p = \left(a^2 r^{n-1}\right)^{\frac{n}{2}} = \left(a.ar^{n-1}\right)^{\frac{n}{2}}$$

$$= (ab)^{\frac{n}{2}} \text{ or } p^2 = (ab)^n$$

Illustration 8:

If a G. P. the first term is 7, the last term 448, and the sum 889; find the common ratio.

Solution:

$a = 7$, $l = ar^{n-1} = 448$

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} = 889$$

Here $S_n = \dfrac{r.\left(a\,r^{n-1}\right) - a}{r-1} = \dfrac{448r - a}{r-1}$

$$\Rightarrow r = 2 \text{ \& } a = 7$$

- **HARMONIC PROGRESSION (H.P.)**

A sequence is said to be in harmonic progression, if and only if the reciprocal of its terms form an arithmetic progression.

For example

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ ... form an H.P., because 2, 4, 6, ... are in A.P.

(a) If a, b are first two terms of an H.P. then

$$t_n = \frac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$$

(b) There is no formula for sum of n terms of an H.P.

(c) If terms are given in H.P. then the terms could be picked up in the following way

- For three terms in H.P, we choose them as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$
- For four terms in H.P, we choose them as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$
- For five terms in H.P, we choose them as $\frac{1}{a-2d}, \frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$

ii) **Useful properties:**

If every term of a H.P. is multiplied or divided by some non zero fixed quantity, the resulting progression is a H.P.

Illustration 9:

If $a_1, a_2, a_3, \ldots a_n$ are in harmonic progression, prove that

$a_1a_2 + a_2a_3 + \ldots + a_{n-1}\ a_n = (n-1)\ a_1a_n$.

Solution:

Since $a_1, a_2, \ldots, a_n$ are in H.P.,

$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in A.P. having common difference d (say) .

$$\therefore \quad \frac{1}{a_2} - \frac{1}{a_1} = d, \frac{1}{a_3} - \frac{1}{a_2} = d, \ldots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d$$

or $a_1 - a_2 = d(a_1a_2),\ a_2 - a_3 = d\ (a_2a_3),\ \ldots,\ (a_{n-1} - a_n) = d(a_{n-1}\ a_n\ )$

Adding the above relations, we get

$a_1 - a_n = d\ (a_1a_2 + a_2a_3 + \ldots + a_{n-1}\ \ a_n)$ ... (1)

Now $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d \quad \therefore \quad \frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$

or $(a_1 - a_n) = (n - 1)\ d\ a_n\ a_1$ ... (2)

Putting the value of $a_1 - a_n$ from (2) in (1), we get

$(n - 1)\ a_n\ a_1 d = d\ (a_1a_2 + a_2a_3 + \ldots + a_{n-1}\ a_n)$

$\therefore \quad (n - 1)\ a_na_1 = a_1a_2 + a_2a_3 + \ldots + a_{n-1}\ \ a_n.$

Illustration 10:

Find H. P. whose 3rd and 14th terms are respectively $\frac{6}{7}$ and $\frac{1}{3}$.

Solution:

Let a & d are first term & common difference of A.P. which is reciprocal of given H.P.

$t_3 = \frac{7}{6} = a + 2d\ \&\ t_{14} = 3 = a + 13d \ \Rightarrow a = \frac{5}{6}\ \&\ d = \frac{1}{6}$

There A.P. is $\frac{5}{6}, 1, \frac{7}{6}, \frac{8}{6}, \frac{9}{6}$.......... &

H.P. is $\frac{6}{5}, 1, \frac{6}{7}, \frac{6}{8}, \frac{6}{9}$..........

Illustration 11:

If the pth, qth and rth terms of a H. P. are a, b, c respectively, prove that $\frac{q-r}{a} + \frac{r-p}{b} + \frac{p-q}{c} = 0$

Solution:

Let T_p, T_q, T_r are pth, qth & rth term of a H.P.

$\frac{1}{T_p} = \frac{1}{a} = A + (p-1)d, \frac{1}{T_q} = \frac{1}{b} = A + (a-1)d$

and $\frac{1}{T_r} = \frac{1}{C} = A + (r-1)d$

Hence $\frac{q-r}{a} + \frac{r-p}{b} + \frac{p-q}{c} = (A+(p-1)d)(q-r) + (A+(q-1)d)(r-p) + (A+(r-1)d)(p-q) = 0$

- **INSERTION OF MEANS BETWEEN TWO NUMBERS**

Let a and b be two given numbers.

(i) Arithmetic Means

- If three terms are in A.P. then the middle term is called the arithmetic mean (A.M.) between the other two i.e. if a, b, c are in A.P. then $b = \frac{a+c}{2}$ is the A.M. of a and b.

- If a, A_1, A_2 , ... A_n, b are in A.P., then A_1, A_2, ... A_n are called n A.M.'s between a and b. If d is the common difference, then $b = a + (n + 2 - 1)d \Rightarrow d = \frac{b-a}{n+1}$

$A_i = a + id = a + i\frac{b-a}{n+1} = \frac{a(n+1-i)+ib}{n+1}$, i = 1, 2, 3, ..., n

Note: The sum of n-A.M′s, i.e., $A_1 + A_2 + ... + A_n = \frac{n}{2}(a+b)$

(ii) Geometric means

- If three terms are in G.P. then the middle term is called the geometric mean (G.M.) between the two. So if a, b, c are in G.P. then $b = \sqrt{ac}$ or $b = -\sqrt{ac}$ corresponding to a & c both are positive or negative respectively.

- If a, G_1, G_2 ... G_n, b are in G.P., then G_1, G_2 ... G_n are called n G.M.s between a and b. If r is the common ratio, then $b = a.r^{n+1} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{(n+1)}}$

$G_i = ar^i = a\left(\frac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}}.b^{\frac{i}{n+1}}$, i = 1, 2, ..., n

Note: The product of n-G. M′s i.e., $G_1 G_2 ... G_n = \left(\sqrt{ab}\right)^n$

(iii) Harmonic mean:

- If a & b are two non-zero numbers, then the harmonic mean of a & b is a number H such that a, H, b are in H.P. & $\frac{1}{H}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ or $H=\frac{2ab}{a+b}$

- If a, H_1, H_2... H_n, b are in H.P., then H_1, H_2 ... H_n are called n H.M.'s between a and b. If d is the common difference of the corresponding A.P., then $\frac{1}{b}=\frac{1}{a}+(n+2-1)d \Rightarrow d=\frac{a-b}{ab(n+1)}$

$$\frac{1}{H_i}=\frac{1}{a}+id=\frac{1}{a}+i\frac{a-b}{ab(n+1)}, H_i=\frac{ab(n+1)}{b(n-i+1)+ia}, \; i=1,2,3,\ldots,n$$

(iv) Term t_{n+1} is the arithmetic, geometric or harmonic mean of t_1 & t_{2n+1} according as the terms t_1, t_{n+1} t_{2n+1} are in A.P., G.P. or H.P. respectively.

Illustration 12:

If A_1, A_2; G_1 , G_2 and H_1 , H_2 be two A.M.s, G.M.s and H.M.s between two quantities 'a' and 'b' then show that $A_1H_2 = A_2H_1 = G_1G_2 = ab$

Solution:

a, A_1 , A_2, b be are in A.P. ... (1)

a, H_1 , H_2 , b are in H.P.

$\therefore \quad \frac{1}{a},\frac{1}{H_1},\frac{1}{H_2},\frac{1}{b}$ are in A.P.

Multiply by ab.

$\therefore \quad b,\frac{ab}{H_1},\frac{ab}{H_2}, a$ are in A.P.

take in reverse order or $a,\frac{ab}{H_2},\frac{ab}{H_1},b$ are in A.P. ... (2)

Compare (1) and (2)

$\therefore \quad A_1=\frac{ab}{H_2}$ and $A_2=\frac{ab}{H_1}$

$\therefore \quad A_1H_2 = A_2H_1 = ab = G_1G_2$

Illustration 13:

Between 2 & 100, 13 means are inserted then find the 9^{th} mean if means are

i) arithmetic ii) geometric iii) harmonic

Solution:

i. a = 2, b = 100, n = 13

$$A_9 = a + 9d = a + 9\frac{b-a}{n+1} = 65$$

ii. a = 2, b = 100, n = 13

$$G_9 = ar^9 = a\left(\frac{b}{a}\right)^{\frac{9}{14}} = 2(50)^{\frac{9}{14}}$$

iii) a = 2, b = 100, n = 13 reciprocal of harmonic is A.P. where $t_1 = \frac{1}{2}$ & $t_{15} = \frac{1}{100}$

$$\frac{1}{H_9} = t_{10} = t_1 + 9d \quad = \frac{1}{2} + 9 \times \frac{-7}{200}$$

Hence $H_9 = \frac{200}{37}$

Illustration 14:

If H.M. & A.M. of two numbers are 3 & 4 respectively, find the numbers.

Solution:

Let numbers are a & b

$$\frac{2ab}{a+b} = 3 \ \& \ \frac{a+b}{2} = 4 \ \Rightarrow ab = 12 \text{ and } a + b = 8$$

solving there we get a = 6, b = 2 or a = 2, b = 6

Illustration 15:

The sum of the two numbers is 6 times their geometric mean. Show that the number are in the ratio $3 + 2\sqrt{2} : 3 - 2\sqrt{2}$.

Solution:

given $a + b = 6\sqrt{ab}$

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 6 \qquad \left(\text{Here } \sqrt{\frac{a}{b}} = t\right)$$

$$\Rightarrow t+\frac{1}{t}=6 \Rightarrow \frac{a}{b}=\left(3+2\sqrt{2}\right)^2=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

Illustration 16:

If b is the harmonic mean between a and c, prove that $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$.

Solution:

$$b=\frac{2ac}{a+c} \Rightarrow \frac{1}{b-a}+\frac{1}{b-c}=\frac{2}{\frac{2ac}{a+c}-a}+\frac{2}{\frac{2ac}{a+c}-c}=\frac{1}{a}+\frac{1}{c}$$

- **ARITHMETICO-GEOMETRIC SERIES**

A series whose each term is formed, by multiplying corresponding terms of an A.P. and a G.P., is called an Arithmetic-geometric series.

e.g. $1+2x+3x^2+4x^3+.....$; $a+(a+d)r+(a+2d)r^2+.....$

(i) Summation of n terms of an Arithmetic-Geometric Series

Let $S_n = a + (a + d) r + (a + 2d)r^2 + ... + [a + (n - 1)d] r^{n-1}$, $d \neq 0$, $r \neq 1$

Multiply by 'r' and rewrite the series in the following way

$$rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + ... + [a + (n - 2)d]r^{n-1} + [a + (n - 1)d]r^n$$

on subtraction,

$$S_n (1 - r) = a + d(r + r^2 + ... + r^{n-1}) - [a + (n-1)d]r^n$$

or, $$S_n(1-r)=a+\frac{dr(1-r^{n-1})}{1-r}-[a+(n-1)d].r^n$$

or, $$S_n=\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]}{1-r}.r^n$$

(ii) Summation of Infinite Series

If $|r| < 1$, then $(n-1)r^n, r^{n-1} \to 0$, as $n \to \infty$.

Thus $S_\infty = S = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$

Illustration 17:

Find the sum of infinity of the series $1+\frac{2.1}{3}+\frac{3.1}{3^2}+\frac{4.1}{3^3}+.......$

Solution:

$$S = 1 + 2.\frac{1}{3} + 3.\frac{1}{3^2} + 4.\frac{1}{3^3} + ...$$

$$\frac{1}{3}S = \frac{1}{3} + 2.\frac{1}{3^2} + 3.\frac{1}{3^3} + ...$$

$$\frac{2}{3}S = 1 + \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + ...\text{upto in infinite}\right) = 1 + \frac{1/3}{1 - \frac{1}{3}} = \frac{3}{2}$$

$$S = \frac{9}{4}$$

Illustration 18:

Find sum to n terms of the series, $1 + 4x + 7x^2 + 10x^3 + ...$ when $|x| < 1$.

Solution:

$T_n = (3n - 2)\, x^{n-1}$

$S_n = 1 + 4x + 7x^2 + 10x^3 + ...+ (3n-2)\, x^{n-1}$ ....(1)

$x\, S_n = x + 4x^2 + 7x^3 + ...+ (3n-5)\, x^{n-1} + (3n-2)x^n$ ....(2)

On subtracting (2) from (1)

$(1 - x)\, S_n = 1 + (3x + 3x^2 + 3x^3 + ...$ upto $(n - 1)$ term$) - (3n-2)\, x^n$

$$(1 - x)S_n = 1 + 3x\frac{\left(1 - x^{n-1}\right)}{1 - x} - (3n - 2)x^n$$

$$S_n = \frac{1}{1 - x}\left(1 + \frac{3x}{1 - x}\left(1 - x^{n-1}\right) - (3n - 2)x^n\right)$$

- **SUM OF MISCELLANEOUS SERIES**

(i) Difference Method

Suppose a_1, a_2, a_3, ... is a sequence such that the sequence $a_2 - a_1, a_3 - a_2$, ... is either an A.P. or G.P. The nth term 'a_n' of this sequence is obtained as follows.

$S = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$

$S = \quad a_1 + a_2 + ... + a_{n-2} + a_{n-1} + a_n$

$$\Rightarrow a_n = a_1 + \left[(a_2 - a_1) + (a_3 - a_2) + ... + (a_n - a_{n-1})\right]$$

Since the terms within the brackets are either in an A.P. or in a G.P. we can find the value of a_n, the nth term. We can now find the sum of the n terms of the sequence as $S = \sum_{k=1}^{n} a_k$

(ii) $V_n - V_{n-1}$ Method

Let T_1, T_2, T_3, ... be the terms of a sequence. If there exists a sequence V_1, V_2, V_3 ... satisfying $T_k = V_k - V_{k-1},\ k \geq 1$,

then $S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (V_k - V_{k-1}) = V_n - V_0$.

Illustration 19 :

Find the sum of n terms of the series 3 + 7 + 14 + 24 + 37 + ...

Solution:

Clearly here the differences between the successive terms are

7 – 3, 14 – 7, 24 – 14, ... i.e., 4, 7, 10, ... which are in A.P.

$\therefore$ $T_n = an^2 + bn + c$

Thus we have 3 = a + b + c, 7 = 4a + 2b + c and 14 = 9a + 3b + c

Solving we get, $a = \frac{3}{2}, b = -\frac{1}{2}, c = 2$. Hence $T_n = \frac{1}{2}(3n^2 - n + 4)$

$\therefore$ $S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n]$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2 + n + 4)$$

Illustration 20 :

Find the sum of n terms of the series 3 + 8 + 22 + 72 + 266 + 1036 + ...

Solution:

1st difference 5, 14, 50, 194, 770, ...

2nd difference 9, 36, 144, 576, ...

They are in G.P. whose nth term is $ar^{n-1} = a4^{n-1}$

$\therefore$ T_n of the given series will be of the form

$T_n = a4^{n-1} + bn + c$

$T_1 = a + b + c = 3$

$T_2 = 4a + 2b + c = 8$

$T_3 = 16a + 3b + c = 22$. Solving we have a = 1, b = 2, c = 0.

$\therefore$ $T_n = 4^{n-1} + 2n$

$\therefore \quad S_n = \Sigma 4^{n-1} + 2\Sigma n = \frac{1}{3}(4^n - 1) + n(n+1)$.

- **INEQUALITIES**

(i) A.M. ≥ G.M. ≥ H.M.

Let $a_1, a_2, ..., a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \frac{a_1 + a_2 + ...+ a_n}{n}$,

$$G = (a_1 a_2 ....a_n)^{1/n} \text{ and } H = \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... \frac{1}{a_n}\right)}$$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = ... = a_n$

(ii) Weighted Means

Let $a_1, a_2, ..., a_n$ be n positive real numbers and $w_1, w_2, ..., w_n$ be n positive rational numbers. Then we define weighted Arithmetic mean (A*), weighted Geometric mean (G*) and weighted harmonic mean (H*) as

$$A^* = \frac{a_1 w_1 + a_2 w_2 + ... + a_n w_n}{w_1 + w_2 + ... + w_n}, \quad G^* = (a_1^{w_1}.a_2^{w_2}...a_n^{w_n})^{\frac{1}{w_1 + w_2 + ... w_n}}$$

$$\text{and } H^* = \frac{w_1 + w_2 + ... + w_n}{\frac{w_1}{a_1} + \frac{w_2}{a_2} + ... + \frac{w_n}{a_n}}.$$

$A^* \geq G^* \geq H^*$ More over equality holds at either place if & only if $a_1 = a_2 i ....=.a_n$

(iii) Cauchy's Schwartz Inequality:

If $a_1, a_2, ...a_n$ and $b_1, b_2,.........., b_n$ are 2n real numbers, then

$(a_1b_1 + a_2b_2 +.............+ a_n b_n)^2 \leq (a_1^2 + a_2^2 +...........+ a_n^2)(b_1^2 + b_2^2 +..........+ b_n^2)$ with the equality holding if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$.

Illustration 21:

Prove that $\left(\frac{a+b}{2}\right)^{a+b} > a^b.b^a, \; a, b \in N; a \neq b.$

Solution:

Let us consider b quantities each equal to a and a quantities each equal to b. Then since A.M. > G.M.

$$\frac{(a+a+a+\ldots b\ \text{times})+(b+b+b+\ldots a\ \text{times})}{a+b} > [(a.a.a\ldots b\ \text{times})\ (b.b.b.\ \ldots\ a\ \text{times})]^{1/(a+b)}$$

$$\Rightarrow \quad \frac{ab+ab}{a+b} > (a^b b^a)^{1/(a+b)} \quad \Rightarrow \quad \frac{2ab}{a+b} > (a^b b^a)^{1/(a+b)}$$

Now $\frac{a+b}{2} > \frac{2ab}{a+b}$ (A.M. > H.M.)

$$\Rightarrow \quad \left(\frac{a+b}{2}\right)^{a+b} > a^b . b^a .$$

- **ARITHMETIC MEAN OF mth POWER**

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers and let m be a real number, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m, \text{ if } m \in R - [0,1].$$

However if $m \in (0, 1)$, then $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} \leq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$

Obviously if $m \in \{0,1\}$, then $\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} = \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$

Illustration 22:

Prove that $a^4 + b^4 + c^4 \geq abc(a + b + c)$, $[a, b, c > 0]$

Solution:

Using mth power inequality, we get

$$\frac{a^4 + b^4 + c^4}{3} \geq \left(\frac{a+b+c}{3}\right)^4$$

$$= \left(\frac{a+b+c}{3}\right)\left(\frac{a+b+c}{3}\right)^3 \geq \left(\frac{a+b+c}{3}\right)[(abc)^{1/3}]^3 \quad (\because \text{ A.M} \geq \text{G.M})$$

or $$\frac{a^4+b^4+c^4}{3} \geq \left(\frac{a+b+c}{3}\right)abc$$

$\therefore$ $a^4 + b^4 + c^4 \geq abc\,(a + b + c)$.

Illustration 23:

Prove that $\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c} \geq \frac{9}{2}$, if $s = a + b + c$, $[a, b, c > 0]$

Solution:

We have to prove that $\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$

for the proof , using mth power theorem of inequality, we get

$$\frac{(a+b)^{-1}+(b+c)^{-1}+(c+a)^{-1}}{3} \geq \left[\frac{a+b+b+c+c+a}{3}\right]^{-1}$$

or, $$\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b} \geq \frac{9}{2(a+b+c)}$$

Aliter :

A.M. $\geq$ H.M.

$$\Rightarrow \frac{(a+b)+(b+c)+(c+a)}{3} \geq \frac{3}{\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}}$$

$$\Rightarrow \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{9}{2(a+b+c)}$$

KEY POINTS

A.P.

If a = first term, d = common difference and n is the number of terms, then

- **nth term is denoted by t_n and is given by** $t_n = a + (n - 1)d$.
- **Sum of first n terms is denoted by S_n and is given by** $S_n = \frac{n}{2}[2a + (n-1)d]$

  or $S_n = \frac{n}{2}(a + l)$, where l = last term in the series i.e., $l = t_n = a + (n - 1)d$.
- **Arithmetic mean A of any two numbers a and b**

  $A = \frac{a+b}{2}$.
- **Sum of first n natural numbers ($\sum n$)**

  $\sum n = \frac{n(n+1)}{2}$, where $n \in N$.
- **Sum of squares of first n natural numbers ($\sum n^2$)**

  $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$
- **Sum of cubes of first n natural numbers** (Σn^3)

  $\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$

G.P.

If a = first term, r = common ratio and n is the number of terms, then

- n^{th} term, denoted by t_n, is given by $t_n = ar^{n-1}$
- Sum of first n terms denoted by S_n is given by $S_n = \frac{a(1-r^n)}{1-r}$ or $\frac{a(r^n-1)}{r-1}$

  In case r = 1, $S_n = na$.
- **Sum of infinite terms (S_∞)**

  $S_\infty = \frac{a}{1-r}$ (for $|r| < 1$ & $r \neq 0$)

H.P.

- If a, b are first two terms of an H.P. then $t_n = \dfrac{1}{\frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)}$
- There is no formula for sum of n terms of an H.P.

MEANS

- If three terms are in A.P. then the middle term is called the arithmetic mean (A.M.) between the other two i.e. if a, b, c are in A.P. then $b = \frac{a+c}{2}$ is the A.M. of a and b.
- If a, A_1, A_2 , ... A_n, b are in A.P., then A_1, A_2, ... A_n are called n A.M.'s between a and b.

  If d is the common difference, then $b = a + (n + 2 - 1)\, d \Rightarrow d = \dfrac{b-a}{n+1}$

  $A_i = a + id = a + i\,\dfrac{b-a}{n+1} = \dfrac{a(n+1-i)+ib}{n+1}$, i = 1, 2, 3, ..., n
- If three terms are in G.P. then the middle term is called the geometric mean (G.M.) between the two. So if a, b, c are in G.P. then $b = \sqrt{ac}$ or $b = -\sqrt{ac}$ corresponding to a & c both are positive or negative respectively.
- If a, G_1, G_2 ... G_n, b are in G.P., then G_1, G_2 ... G_n are called n G.M.s between a and b. If r is the common ratio, then $b = a.r^{n+1} \Rightarrow r = \left(\dfrac{b}{a}\right)^{\frac{1}{(n+1)}}$

  $G_i = ar^i = a\left(\dfrac{b}{a}\right)^{\frac{i}{n+1}} = a^{\frac{n+1-i}{n+1}}.b^{\frac{i}{n+1}}$, i = 1, 2, ..., n
- If a & b are two non-zero numbers, then the harmonic mean of a & b is a number H such that a, H, b are in H.P. & $\dfrac{1}{H} = \dfrac{1}{2}\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ or $H = \dfrac{2ab}{a+b}$
- If a, H_1, H_2... H_n, b are in H.P., then H_1, H_2 ... H_n are called n H.M.'s between a and b. If d is the common difference of the corresponding A.P., then

  $\dfrac{1}{b} = \dfrac{1}{a} + (n+2-1)\,d \Rightarrow d = \dfrac{a-b}{ab(n+1)}$

  $\dfrac{1}{H_i} = \dfrac{1}{a} + id = \dfrac{1}{a} + i\,\dfrac{a-b}{ab(n+1)}$; $H_i = \dfrac{ab(n+1)}{b(n-i+1)+ia}$, i = 1, 2, 3, ..., n